Mathematics

Higher

Unit 1

Number

Adding Whole Numbers

When adding whole numbers, we need to line up the digits at the right-hand side, ones in the ones column, tens in the tens column, etc.

Example: 145 + 28

```
  145       145       145
+  28     +  28     +  28
    3        73       173
   1        1        1
```

So 145 + 28 = **173**.

Subtracting Whole Numbers

When subtracting whole numbers, we need to line up the digits at the right-hand side, ones in the ones column, tens in the tens column, etc. If the number we are subtracting from is smaller than the number we have, then we will need to "borrow" from the next number.

Example: 364 – 128

```
   5 1       5 1       5 1
  3 6 4     3 6 4     3 6 4
– 1 2 8   – 1 2 8   – 1 2 8
      6       3 6     2 3 6
```

So 364 – 128 = **236**.

Dividing Whole Numbers:

Example – short division: 3144 ÷ 8

Using short division:

1. 8 doesn't go into 3, so look at the first two digits.
2. 8 goes into 31 three times, with remainder 7.
3. 8 goes into 74 nine times, with remainder 2.
4. 8 goes into 24 three times exactly.

```
   0 3
8)3 1 ⁷4 4

   0 3 9
8)3 1 ⁷4 ²4

   0 3 9 3
8)3 1 ⁷4 ²4
```

So 3144 ÷ 8 = **393**.

Example – long division: 782 ÷ 34

```
       23    (answer line)
34    782
   -  680    (34 x 20 = 680, put 2 in the tens column on the answer line)
      102
   -  102    (34 x 3 = 102, put 3 in the units column on the answer line)
      000
```

Therefore 782 ÷ 34 = 23

Mathematics

Higher

Unit 1

Multiplying Whole Numbers

Note: 12 x 3 is the same as 3 x 12

Example 1 – short multiplication: 127 x 6

```
  1 2 7 ×
      6
  -----
  7 6 2
  -----
  1 4
```

1st work out 7 × 6 = 42
Write down the 2 units in the box and carry the 4 tens under the tens column

Next work out 2 × 6 = 12 (or 20 × 6) you need to add on the carry of 4 (or 40) to make 16 (or 160). Place 6 (or 6 tens) in the box in the tens column and carry the 1(or 100) in the hundreds column

Lastly, work out 1 × 6 (or 100 × 6) and add on the carry of 1 (or 100) to make 7 (or 700). Place 7 in the box in the hundreds column

Example 2 – long multiplication: 352 x 27

Method 1: Column Method

```
    3 5 2 ×
      2 7
  -------
  2 4 6 4 +
  7 0 4 0
  -------
  9 5 0 4
  1
```

352 x 7 = 2464

Put a 0 in the ones column as we are now multiplying by the number in the tens column
352 x 2 = 704

Add the two rows,
2464 + 7040 = 950

Example 3 – long multiplication: 23 x 34

Method 2: Grid Method

23 is split into 20 and 3

34 is split into 30 and 4

x	20	3
30	600	90
4	80	12

30 x 20 = 600

30 x 3 = 90

4 x 20 = 80

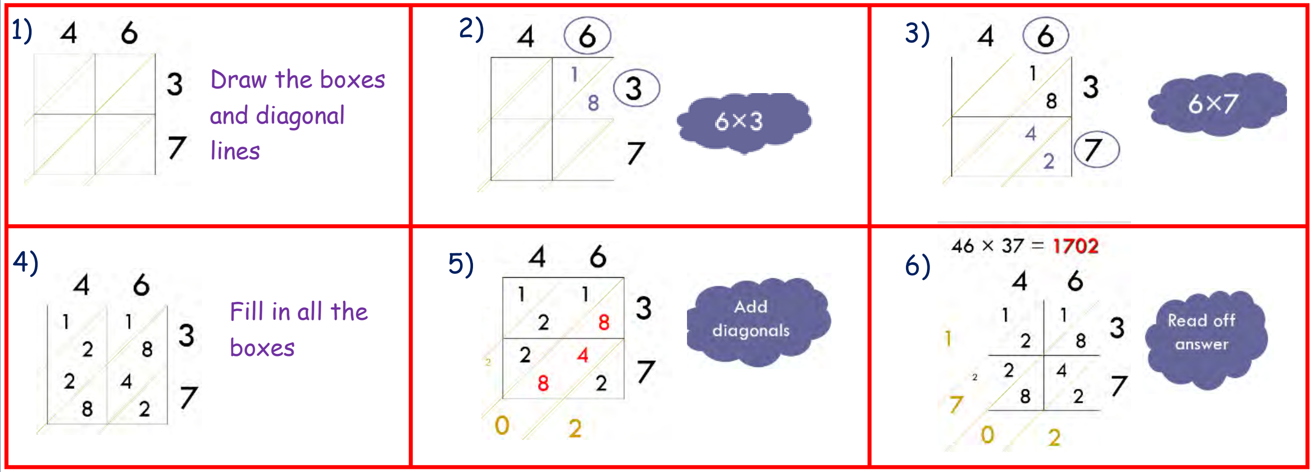
4 x 3 = 12

Add all the answers

600 + 90 + 80 + 12 = 782

Example 4 – long multiplication: 46 x 37

Method 3: Box Method



Mathematics

Higher

Unit 1

Multiplying and Dividing by Multiples of 10:

When you multiply your digits move left, and when you divide your digits move right. The distance they move depends on the amount of zeros in your number (10, 100, 1000 …). Eg. If you are multiplying by 100 the digits move to the left 2 places because 100 has 2 zeros.

Example 1: 43 x 10 = 430

43 moves one place value to the left (10 has one zero) and the space is filled in with a zero

Example 2: 789 x 1000 = 789000

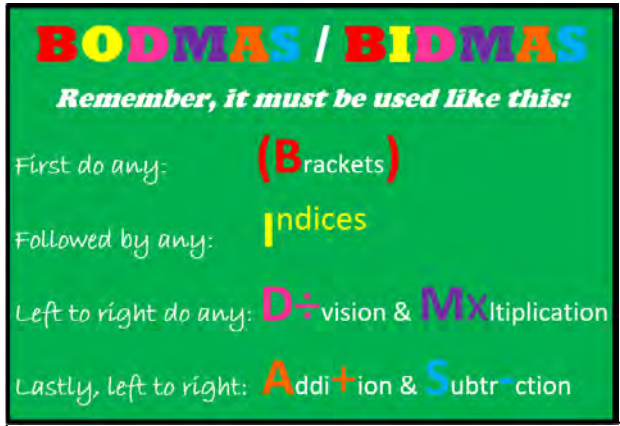
789 moves three place values to the left (1000 has three zeros) and the spaces are filled in with zeros

Example 3: 3200 ÷ 100 = 32

3200 moves two place values to the right (100 has two zeros),

Example 4: 86 ÷ 10 = 8.6

86 moves one place value to the right (10 has one zero), the 6 moves into the tenths column, the answer is a decimal



BIDMAS / BODMAS:

BIDMAS or BODMAS is a way of helping you to remember the order in which to do your calculations.

Example 1: 2 + 7 x 10

= 2 + 70

= 72

This question involves addition and multiplication, using the rules of BIDMAS, multiplication is first

Example 2: 22 – 6 + 4

= 16 + 4

= 20

This question involves subtraction and addition, using the rules of BIDMAS, work left to right doing whatever is first

Example 3: $(4 + 5)^2 - 4 \times 9$

$= (9)^2 - 4 \times 9$

$= 81 - 4 \times 9$

$= 81 - 36$

$= 45$

This question involves multiple operations, just follow the rules of BIDMAS. Brackets first, then indices, then multiplying, then subtraction

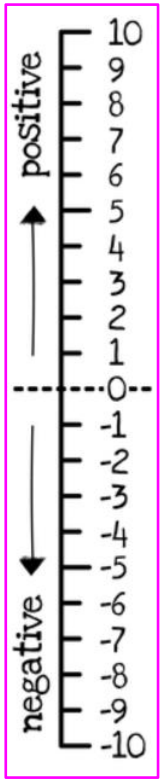
Mathematics

Higher

Unit 1

Negative Numbers

Negative numbers can be represented on a number line.
You will notice that -3 is higher than -7.
It is possible to have negative temperatures when it is very cold (-3°C).



Addition and Subtraction of Negative Numbers

For addition we move up the number line.

For subtraction we move down the number line.

When we have two signs (+ or -) immediately next to each other, we change the 2 signs into 1 using the following rules	+ + → + − − → + + − → − − + → −

Example 1: -5 + 3 = -2 This means we start at -5 and move up 3 places

Example 2: 2 – 6 = -4 This means we start at 2 and move down 6 places

Example 3: 4 + - 3 = 4 - 3 = 1 (+ - → −)

Example 4: -8 - - 6 = -8 + 6 = -2 (- - → +)

Multiplying and Dividing Negative Numbers

Use the following rules:

Positive(+)	×/ ÷	Positive(+)	gives a Positive(+) answer
Negative(-)	×/ ÷	Negative(-)	gives a Positive(+) answer
Positive(+)	×/ ÷	Negative(-)	gives a Negative(-) answer
Negative(-)	×/ ÷	Positive(+)	gives a Negative(-) answer

Example 1: -7 x -3 = 21 (A negative x a negative = a positive)

Example 2: 18 ÷ -6 = -3 (A positive ÷ a negative = a negative)

Ordering Directed Numbers

Think of a number line, which number would be further down the number line? Which number would be higher up?

Example 1: Put the following in ascending order (Smallest to biggest)

12, 0, 23, -21, -17, -3

-21, -17, -3, 0, 12, 23

Example 2: Put the following in descending order (Biggest to smallest)

-97, 85, 51, 2, -6, -47

85, 51, 2, -6, -47, -97

Mathematics

Higher

Unit 2

Types of Number and Use of Index Notation

Types of Number

Square Numbers

You can get a Square Number by multiplying any whole number (integer) by itself

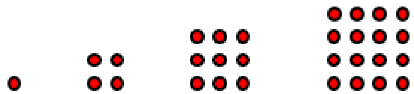
So: The first square number is 1, because 1 x 1 = 1.

The second square number is 4, because 2 x 2 = 4, and so on…

The first ten square numbers are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

Note:

You can also get all the square numbers by counting the dots in square patters:



Prime Numbers

A prime number is a number that is only divisible by itself and 1; a prime number has exactly 2 factors.

For example: 7 is a prime number as it has two factors (1 and 7),

21 is NOT a prime number as it has four factors (1, 3, 7 and 21)

Note: 1 is **NOT** a prime number, as it only has one factor (1)

2 is the only **even** prime number as it has two factors (1 and 2)

The first ten prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

Cube Numbers

You can get a Cube Number by multiplying any whole number (integer) by itself and then by itself again.

So: The first cube number is 1, because 1 x 1 x 1 = 1.

The second cube number is 8, because 2 x 2 x 2 = 8, and so on…

The first five cube numbers are: 1, 8, 27, 64, 125.

Factors

The Factors of a number are all the whole numbers (integers) that divide into your number exactly (there must not be a remainder).

For example: the factors of 12 are: 1, 2, 3, 4, 6 and 12, the factors of 55 are: 1, 5, 11, and 55

Note: 1 is a factor of all numbers, and so is the number itself.

Multiples

The Multiples of a number are all the numbers in the number's times table.

For example: the multiples of 2 are all the numbers in the 2 times table (2, 4, 6, 8, 10, …), the first three multiples of 6 are 6, 12, 18.

Mathematics

Higher

Unit 2

Indices

Key Words:

Indices are just another word for "power"

The base → 5^4 ← The index or power

Two things you must remember about indices:

- Indices only apply to the number or letter they are to the right of – the base
  e.g. in abc^2, the squared only applies to the c, and nothing else. If you wanted the squared to apply to each term, it would need to be written as $(abc)^2$.
- Indices do not mean multiply
  e.g. 6^3 does not mean 6 x 3, it means 6 x 6 x 6

Multiplying Indices

Using index notation: $a^m \times a^n = a^{m+n}$

What it means: Whenever you are multiplying two terms with the same base, you can just add the powers!

Numbers: If there are numbers IN FRONT of your bases, then you must multiply those numbers together as normal

Examples

$x^3 \times x^4 = x^7$ ✓ Classic wrong answer: x^{12} ✗

$2^5 \times 2^3 = 2^8$ ✓ Classic wrong answer: 4^8 ✗

$3p^4 \times 2p^5 = 6p^9$ ✓ Classic wrong answer: $6p^{20}$ ✗

$2ab^2c \times 5ab^2c^3 = 10a^2b^4c^4$ ✓

Remember: if a base does not appear to have a power, the power is a 1.

e.g. $2ab^2c = 2a^1b^2c^1$

Dividing Indices

Using index notation: $a^m \div a^n = a^{m-n}$ Or $\frac{a^m}{a^n} = a^{m-n}$

What it means: Whenever you are dividing two terms with the same base, you can just subtract the powers!

Numbers: If there are numbers IN FRONT of your bases, then you must divide those numbers as normal

Examples

$x^{12} \div x^4 = x^8$ ✓ Classic wrong answer: x^3 ✗

$\frac{5^7}{5^3} = 5^4$ ✓ Classic wrong answer: 1^4 ✗

$\frac{20k^{10}}{5k^5} = 4k^5$ ✓ Classic wrong answer: $4k^2$ ✗

Mathematics

Higher

Unit 2

A Power of a Power

Using index notation: $(a^m)^n = a^{m \times n}$

What it means: Whenever you have a base and it's power raised to another power, you simply multiply the powers together but keep the base the same!

Numbers: If there is a number IN FRONT of the base, then you must raise that number to the power

Examples

$(x^5)^3 = x^{15}$ ✓ Classic wrong answer: x^8 ✗

$(2^3)^2 = 2^6$ ✓ Classic wrong answer: 4^6 ✗

$(3a^4)^3 = 27a^{12}$ ✓ Classic wrong answer: $9a^{12}$ ✗

$(2a^3b^2c)^5 = 32a^{15}b^{10}c^5$ ✓

Zero Index

Using index notation: $a^0 = 1$

What it means: Anything to the power of zero is 1!

Examples $x^0 = 1$ $17^0 = 1$ $5x^0 = 5 \times 1 = 5$

Fractional Indices

Using index notation: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

What it means: When a power is a fraction it means you take the root of the base and which root you take depends on the number on the denominator of the fraction, the numerator of the fraction is the power you raise the answer to.

Examples

$64^{\frac{1}{2}} = \sqrt{64} = 8$ The power of a half means the square root

$27^{\frac{1}{3}} = \sqrt[3]{27} = 3$ The power of a third means the cube root

$32^{\frac{1}{5}} = \sqrt[5]{32} = 2$ Because $2^5 = 32$

When the numerator is 1 you do not need to raise the answer to a power ($\sqrt{64}$ is the same as $(\sqrt{64})^1$).

$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

Negative Indices

Using index notation: $a^{-m} = \frac{1}{a^m}$

What it means: A negative sign in front of a power is the same as writing "one divided by the base and power". This is called the RECIPROCAL

Note: Only the power and base are flipped over, nothing else!

Examples $x^{-2} = \frac{1}{x^2}$ $5^{-4} = \frac{1}{5^4}$ $5a^{-3} = \frac{5}{a^3}$

$\left(\frac{1}{3}\right)^{-1} = \left(\frac{3}{1}\right)^1 = 3$ $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2 = 16$ $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

Negative Fractional Indices

Using index notation: $a^{-\frac{m}{n}} = (\sqrt[n]{a})^{-m} = \frac{1}{(\sqrt[n]{a})^m}$

What it means: This is just a mix of the negative indices rule and the fractional indices rule

Examples

$16^{-\frac{1}{2}} = (\sqrt{16})^{-1} = \frac{1}{\sqrt{16}} = \frac{1}{4}$

$8^{-\frac{2}{3}} = (\sqrt[3]{8})^{-2} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

Mathematics

Higher

Unit 2

Prime Factors

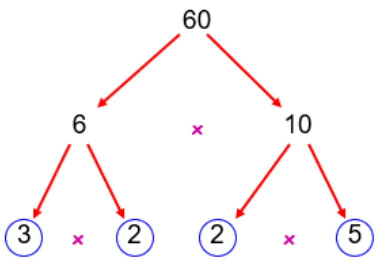
Prime Factors

Any positive integer can be written as a product of its prime factors.

Now, that may sound complicated, but all it means is that you can break up any number into a multiplication of prime numbers, and it's really easy to do with Factor Trees!

Don't Forget: 1 is NOT a prime number, so will NEVER be in your factor tree

Example: Express 60 as a product of its prime factors



3 x 2 x 2 x 5 = 60

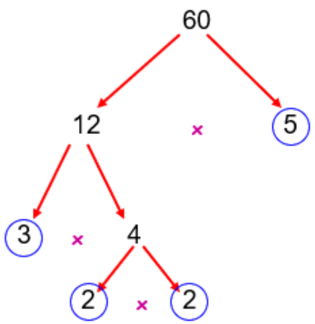
You can break the number up however you like:

6 x 10 or 12 x 5

Continue breaking up each new number into a multiplication

Stop when you reach a Prime Number and put a circle around it

Check your answer by multiplying all the numbers together



3 x 2 x 2 x 5 = 60

Note: Even though we started a different way, we still ended up with the same answer.

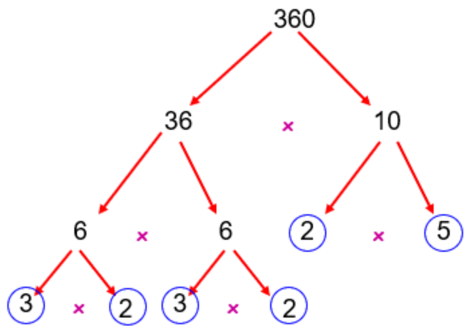
Try writing your answer starting with the smallest numbers:

$60 = 2 \times 2 \times 3 \times 5$

Then write the answer using indices:

$60 = 2^2 \times 3 \times 5$

e.g. Express 360 as a product of its prime factors



3 x 2 x 3 x 2 x 2 x 5 = 360

360 = 2 x 2 x 2 x 3 x 3 x 5

$360 = 2^3 \times 3^2 \times 5$

You can break the number up however you like. 36 x 10 is just easy to spot

Continue breaking up each new number into a multiplication

Stop when you reach a Prime Number and put a circle around it

Check your answer by multiplying all the numbers together

Write the numbers in order

If you can, use indices

Or you can try this 'ladder' method:

On the left:
Start with the given number.
All the other numbers are answers to the division from the right.
E.g. 60 ÷ 2 = 30
30 ÷ 2 = 15

60	2
30	2
15	3
5	5
1	

On the right:
All the numbers are **prime numbers** that go into the numbers on the left.
You divide by that prime number and write the answer on the left.
Continue this until you get to 1.

Check your answer:
Multiply together the numbers on the right:

2 x 2 x 2 x 5 = 60

Mathematics

Higher

Unit 2

Highest Common Factor, Lowest Common Multiple and Perfect Squares

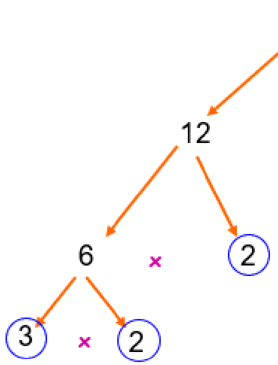
The Highest Common Factor (HCF) of two numbers, is the highest number that divides exactly into both numbers.

The Lowest Common Multiple (LCM) of two numbers, is the lowest number that is in the times table of both numbers.

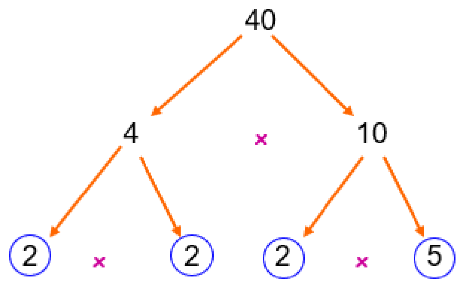
Finding the Highest Common Factor and Lowest Common Multiple Using Prime Factors.

Example: Find the LCM and HCF of 24 and 40

First, use Factor Trees to express your numbers as products of their prime factors:



24 = 2 x 2 x 2 x 3



40 = 2 x 2 x 2 x 5

Write out your answers:

24 = 2 x 2 x 2 x 3
40 = 2 x 2 x 2 x 5

Then in index form, like this

$24 = 2^3 \times 3$
$40 = 2^3 \times 5$

To get the **Highest Common Factor**, multiply together the numbers that appear in **BOTH** lists.

HCF = $2^3 = \mathbf{8}$

To get the **Lowest Common Multiple**, multiply together all the numbers that appear in either list, taking the highest power seen for each one. **Do not** include any duplicates.

LCM = $2^3 \times 3 \times 5 = \mathbf{120}$

Perfect Squares/Square Numbers

Example 1: $60 = 2^2 \times 3 \times 5$, is 60 a perfect square? If not, what do you need to multiply 60 by to make it a perfect square?

60 is not a perfect square as the indices on the prime factorisation are not all even numbers

2 is even → $2^2 \times 3 \times 5$ ← Remember, if you can't see an index number it means it is a 1, 1 is not an even number

To make 60 a perfect square we need to multiply it by:

$$3 \times 5 = 15$$

(This would then make all the indices even $2^2 \times 3^2 \times 5^2$)

Example 2: $400 = 2^4 \times 5^2$, is 400 a perfect square?

400 is a perfect square as the indices on the prime factorisation are all even numbers

4 is even → $2^4 \times 5^2$ ← 2 is even

Example 3: The number 32,768 is equal to 2^{15}. Explain how this tells you that 32,768 is not a square number.

The index number, 15, is not an even number, so 32,768 is not a square number.

Mathematics

Higher

Unit 3

Decimals

Place Value and Ordering Decimals

Place value is the value given to a digit by its place in a number.

Ascending means smallest to biggest; descending means biggest to smallest

Decimal Place Value Table

Ten thousands	Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths	Ten thousandths

Example:

a) What is the value of the 9 in the number 10.609?
b) What is the value of the 7 in the number 234.75?

Using the place value table:

Ten thousands	Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths	Ten thousandths
			1	0	.	6	0	9	
		2	3	4	.	7	5		

a) 9 thousandths
b) 7 tenths

Ordering Decimal Numbers

Example: Put the following numbers in ascending order 43.85, 43.8, 43.856

Use the place value table to compare the numbers:

Ten thousands	Thousands	Hundreds	Tens	Ones	.	Tenths	Hundredths	Thousandths	Ten thousandths
			4	3	.	8	5	0	
			4	3	.	8	0	0	
			4	3	.	8	5	6	

Ascending means smallest to biggest, so we need the smallest number first.
All the whole numbers are of equal value so we need to start by looking at the decimal places. We can fill any gaps in with zeros to make comparing easier.
Looking at the tenths column, these digits are all the same. Looking at the hundredths column, the 0 is the smallest digit so 43.8 is the smallest number. The other 2 digits are the same so we look at the thousandths column. The zero is the smallest digit here so 43.85 is the next biggest number.

The order is: 43.8, 43.85, 43.856

Mathematics

Higher

Unit 3

Adding and Subtracting Decimals

When we add or subtract decimals, we write the numbers out on top of each other, making sure we line up the decimal points.

Adding decimals 4.53 + 1.6

Line up the decimal points

Fill any gaps with zeros

```
  4.53      4.53      4.53
+ 1.60    + 1.60    + 1.60
   . 3      ₁.13     6₁.13
```

So 4.53 + 1.6 = **6.13**

Add the numbers, moving the decimal point in line to the answer

Example: 0.14 + 8 + 23.7

Step 1 and 2: Line up the decimal points, remember a whole number has an invisible decimal point after it, so 8 is the same as 8.0. Fill in any gaps with zeros

Decimal points lined up

```
  0 . 1 4
  8 . 0 0 +
2 3 . 7 0
---------
---------
```

Step 3: Add the numbers, putting the decimal point in line in the answer

```
  0 . 1 4
  8 . 0 0 +
2 3 . 7 0
---------
3 1 . 8 4
---------
1
```

Subtracting decimals 8.5 – 3.07

Fill any gaps with zeros

Line up the decimal points

```
  8.⁴5¹0      8.⁴5¹0      8.⁴5¹0
– 3.07      – 3.07      – 3.07
   . 3         .43        5.43
```

So 8.5 – 3.07 = **5.43**

Subtract the numbers, moving the decimal point in line to the answer

Example: 47 – 19.43

Step 1 and 2: Line up the decimal points, remember a whole number has an invisible decimal point after it, so 47 is the same as 47.0. Fill in any gaps with zeros

Decimal points lined up

```
4 7 . 0 0 -
1 9 . 4 3
---------
---------
```

Step 3: Subtract the numbers, putting the decimal point in line in the answer

```
³4 ¹⁶7 . ⁹0 ¹0 -
1 9 . 4 3
---------
2 7 . 5 7
---------
```

Mathematics

Higher

Unit 3

Multiplying Decimals

When we multiply decimals, we ignore the decimal point and just multiply the numbers, then we count how many decimal places (numbers after the decimal point) there are in the question and put the decimal point back in the answer making sure we have the same number of decimal places in our answer.

Multiplying Decimals 0.03 x 1.1

Ignore the decimal points and multiply the numbers

3 x 11 = 33

(0.03 x 1.1)

So: 0.03 x 11 = 0.033

Count how many decimal places there are in the question (3 decimal places altogether)

Count backwards from the end of the answer the same number of decimal places, put in the decimal point

Example 1: 8 x 0.5

Step 1: Ignore the decimal points and multiply the numbers 8 x 5

8 x 5 = 40

Step 2: Count the number of decimal places in the question

8 x 0.5 One decimal place

Step 3: Count backwards from the end of the answer the same number of decimal places, put in the decimal point.

4 . 0

So, 8 x 0.5 = 4 (which is the same as 4.0)

Example 2: 2.6 x 2.3

Step 1: Ignore the decimal points and multiply the numbers 26 x 23 (This is a long multiplication, look back at unit 1 if you are unsure how to do it)

```
   2 6 x
   2 3
  -----
   7 8 +
 5 2 0
  -----
 5 9 8
```

Step 2: Count the number of decimal places in the question

2.6 x 2.3 Two decimal places

Step 3: Count backwards from the end of the answer the same number of decimal places, put in the decimal point.

5 . 9 8

So, 2.6 x 2.3 = 5.98

Mathematics

Higher

Unit 3

Dividing Decimals

When we divide a decimal by a whole number (e.g. 18.6 ÷ 3) we can set it up like a normal dividing question using the bus stop method, moving the decimal point in line up into the answer.

When we divide a whole number by a decimal (e.g. 25 ÷ 0.5), or a decimal by a decimal (e.g. 18.6 ÷ 0.03) we need to make the number we are dividing by into a whole number first, then divide in the usual way.

Dividing Decimals by a whole number   2.52 ÷ 4

Set up using the bus stop method, divide as usual, the decimal point moves in line up into the answer

0 . 6 3

4 | 2 . ^{2}5 ^{1}2

Example: 12.84 ÷ 6

Set up using the bus stop method, divide as usual, the decimal point moves in line up into the answer

0 2 . 1 4

6 | 1 ^{1}2 . 8 ^{2}4

Dividing by a Decimal   4.06 ÷ 0.2

We do not like to divide by a decimal, so we make it a whole number first. We can do this by either multiplying by 10, 100, 100 …

Remember, whatever we multiply the one number by we must also multiply the other by as well.

Also multiply by 10, must do the same to both sides

4.06 x 10 = 40.6   0.2 x 10 = 2

We are now working out 40.6 ÷ 2

2 0 . 3

2 | 4 0 . 6

So, 4.06 ÷ 0.2 = 20.3

Multiply by 10 to make 0.2 a whole number

Answer to 4.06 ÷ 0.2 is the same as the answer to 40.6 ÷ 2

Example: 45 ÷ 0.04

Step 1: Multiply both sides by 100 to make 0.04 a whole number

45 x 100 = 4500   0.04 x 100 = 4

We are now working out 4500 ÷ 4

Step 2: Set up using the bus stop method, divide as usual

1 1 2 5

4 | 4 5 ^{1}0 ^{2}0

So, 45 ÷ 0.04 = 1125

Mathematics

Higher

Unit 4

Round to an Appropriate Degree of Accuracy

There are lots of degrees of accuracy you will need to know how to round to, but the way to work out any rounding question is always the same:

Step 1: Circle the last digit you need – what I will call the Key Digit

Step 2: Look at the unwanted digit to the right to it – if it is 5 or above add one on to your Key Digit, if it is less than five, leave your Key Digit alone.

Step 3: Be very careful of the dreaded number 9…

Rounding to Nearest Whole Number, 10, 100, 1000 etc

Remember: the size of your rounded number should be a similar size to the number in the question, and you must use zeros to help you with this.

Example 1

Round 3.825 to the nearest whole number

(3). 8 2 5

1. Our Key Digit is always the degree of accuracy the question asks for, which in this case is whole numbers, so we need the 3.

2. The unwanted digit to the right of it is 8, which is more than 5, so we add one to our Key Digit.

3. So, to the nearest whole number, our answer is:

4

Example 2

Round 4,365,901 to the nearest thousand

4 3 6 (5) 9 0 1

1. We want the nearest thousand, so our Key Digit must be the number that represents the thousands which is the 5

2. The unwanted digit to the right of it is 9, which is more than 5, so we add one to our Key Digit.

3. So, to the nearest whole number, our answer is:

4,365,000

Example 3

Round 3,999 to the nearest ten

3 9 (9) 9

1. We want the nearest ten, so the Key Digit must be the 9 in the tens column

2. The unwanted digit to the right of it is a 9, so we add one on, but we then need to add one on the next 9, and then the 3.

3. So, to the nearest ten, our answer is:

4,000

Mathematics

Higher

Unit 4

Rounding to Decimal Places

You will be asked to round to a given number of decimal places, this can be written as d.p.

E.g. 5.95783… rounded to 2 d.p. is 5.96

Remember, if the question asks for two decimal places, you must give two, no more, no less.

Example 1

Round 5.639 to 1dp

5 . (6) 3 9

1. We start by putting a ring around our Key Digit. The question has asked for 1 decimal place, so our key digit is the 6, as it occupies the 1st decimal place

2. Next we look at the digit to the right to it - the unwanted number 3. It is less than 5, so we leave the key digit alone.

3. So, to one decimal place, our answer is:

5.6

Example 2

Round 12.0482 to 2dp

1 2 . 0 (4) 8 2

1. This time the Key Digit is in the 2nd decimal place, which makes it the 4

2. The unwanted digit to the right of it is an 8, which is more than 5, so we must add one onto our Key Digit

3. So, to two decimal places, our answer is:

12.05

Example 3

Round 25.72037 to 3dp

2 5 . 7 2 (0) 3 7

1. This time the Key Digit is in the 3rd decimal place, which makes it the 0

2. The unwanted digit to the right of it is 3, which is less than 5, so just leave our Key Digit alone

3. So, to three decimal places, our answer is:

25.720

Be careful: The answer is not 25.72, as we must have the 3 decimal places.

Example 4

Round 3.7952 to 2dp

3 . 7 (9) 5 2

1. This time the Key Digit is in the 2nd decimal place, which makes it the 9

2. The unwanted digit to the right of it is a 5, which is 5 or above, so we must add one onto our Key Digit

But: if we add one to our key digit, we get 10. So, we must add one to the next digit as well, which is the 7

3. So, to two decimal places, our answer is:

3.80

Mathematics

Higher

Unit 4

Rounding to Significant Figures

You will be asked to round to a given number of significant figures, this can be written as s.f.

E.g. 59,578 rounded to 3 s.f. is 59,600

Note: The first significant figure is always the first non-zero digit you come across.

Remember: the size of your rounded number should be a similar size to the question, and you must use zeros to help you with this.

Example 1

Round 28.53 to 1 s.f.

(2)8 . 5 3

1. The Key Digit is the first significant figure, which must be the 2, as it is the first non-zero number

2. Look to the number to the right, which is an 8, so we add one on.

3. So, keeping the size of the answer the same as the question with a zero, to 1 s.f. the answer is:

30

Example 2

Round 5,322 to 2 s.f.

5(3)2 2

1. The Key Digit is in the place of the 2nd significant figure, which is the 3

2. The unwanted digit to the right of it is 2, which is less than 5, so we leave our Key Digit alone

3. Again using zeros to help us, to two s.f. the answer is:

5300

Example 3

Round 0.027 to 1 s.f.

0 . 0(2)7

1. Our first significant figure is the first non-zero number, which means it's the 2

2. The unwanted digit to the right of it is 7, so we add one to our Key Digit.

3. No need for extra zeros here, so to 1 s.f. the answer is:

0.03

Example 4

Round 4.0004 to 2 s.f.

4 .(0)0 0 4

1. The 1st sig fig is the 4, and so the 2nd is the 0 (it is after the 4, so it's significant).

2. The unwanted digit to the right of it is 0, which is less than 5, so we leave our Key Digit alone

3. So, to 2 s.f. the answer is:

4.0

Mathematics

Higher

Unit 4

Estimating

When we estimate we round each number to one significant figure to make the calculations easier to do.

E.g. 231 x 8.9

200 x 9 = 1800

So, the actual answer is approximately 1800.

Estimate the value of $\dfrac{6.0602^2}{3.1092 \times 5.95}$

Step 1: First round each value to 1 significant figure: → $\dfrac{6^2}{3 \times 6}$

Step 2: Apply the rules of BIDMAS/BODMAS to calculate the answer: → $\dfrac{6^2}{3 \times 6}$ → $\dfrac{36}{3 \times 6}$ → $\dfrac{36}{18}$ → 2

The actual answer on the calculator is close: 1.98521838...

We can say that: $\dfrac{6.0602^2}{3.1092 \times 5.95} \approx 2$

approximately equal to

Example 1:

Estimate the value of

$$\frac{1.432 \times 62.34}{32.123}$$

Rounded to 1 s.f. $\approx \dfrac{1 \times 60}{30}$

$\approx \dfrac{60}{030}$

≈ 2

Example 2:

Estimate the value of

$$\frac{2.09^3 \times \sqrt{98}}{0.196}$$

Rounded to 1 s.f. $\approx \dfrac{2^3 \times \sqrt{100}}{0.2}$

$\approx \dfrac{8 \times 10}{0.2}$

$\approx \dfrac{80}{0.2}$

≈ 400

Mathematics

Higher

Unit 5

Fractions

A **fraction** is part of a whole, made up of a numerator and a denominator $\frac{2}{3}$ ← Numerator (2), ← Denominator (3)

An **improper fraction** is a fraction where the numerator is greater than the denominator $\frac{18}{7}$ ← Numerator greater than denominator

A **mixed number** is a number made up of a whole number and a fraction $6\frac{4}{9}$ (Whole number: 6; Fraction: $\frac{4}{9}$)

Equivalent Fractions

Equivalent fractions may look different, but they have the same value.

Example 1: $\frac{2}{4} = \frac{1}{2}$ Two quarters are the same as one half, they are equivalent fractions

Example 2: $\frac{1}{4} = \frac{2}{8} = \frac{25}{100}$

Simplifying Fractions

We can make fractions simpler, by dividing the numerator and the denominator by a common factor. (Questions may ask you to "simplify your answer").

Some fractions may simplify more than once, you need to keep simplifying until the fraction cannot be simplified any further.

Example 1: Simplify $\frac{10}{40}$

$$\frac{10}{40} \overset{\div 10}{=} \frac{1}{4}$$

÷ 10 (numerator), ÷ 10 (denominator)

Both 10 and 40 have been divided by 10 to make 1 and 4

Example 2: Simplify $\frac{24}{108}$

$$\frac{24}{108} = \frac{12}{54} = \frac{6}{27} = \frac{2}{9}$$

÷ 2, ÷ 2, ÷ 3 (numerator and denominator)

Simplified in 3 steps

OR

$$\frac{24}{108} = \frac{2}{9}$$

÷ 12 (numerator and denominator)

Simplified in 1 step

Same answer

Mathematics

Higher

Unit 5

Ordering Fractions

To be able to order fractions, they need to have the same denominator first.

Example: Put the following fractions in ascending order (smallest to biggest).

$$\frac{3}{4}, \frac{1}{2}, \frac{5}{6}, \frac{2}{3}$$

Step 1: Find the lowest common multiple of all the denominators

Lowest common multiple of 4, 2, 6, and 3 is 12

Step 2: Make equivalent fractions using the lowest common multiple as the denominator

$$\frac{3}{4} = \frac{9}{12}, \quad \frac{1}{2} = \frac{6}{12}, \quad \frac{5}{6} = \frac{10}{12}, \quad \frac{2}{3} = \frac{8}{12}$$

Step 3: Order the fractions, replace with original fractions

Smallest to biggest $\frac{6}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}$

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$$

Mixed Numbers to Improper Fractions

We can convert a mixed number, e.g. $2\frac{1}{3}$, to an improper fraction, e.g. $\frac{7}{3}$,

Example: Convert $2\frac{4}{7}$ to an improper fraction

Rule: $\frac{\text{(denominator x whole number) + numerator}}{\text{denominator}}$

So, $2\frac{4}{7} = \frac{(7 \times 2) + 4}{7} = \frac{14 + 4}{7} = \frac{18}{7}$

$$2\frac{4}{7} = \frac{18}{7}$$

Improper Fractions to Mixed Numbers

We can convert an improper fraction, e.g. $\frac{7}{3}$, to a mixed number, e.g. $2\frac{1}{3}$.

Example: Convert $\frac{13}{5}$ to a mixed number

$\frac{13}{5}$ means 13 ÷5.

How many 5's are in 13? 2 (this becomes the whole number of our mixed number)

What is the remainder? 3 (this becomes the numerator of the fraction part of our mixed number)

How many 5's are in 13

So, $\frac{13}{5} = 2\frac{3}{5}$

Remainder

Denominator stays the same

Mathematics

Higher

Unit 5

Increasing/Decreasing by a Fraction

To find a fractional increase, first find the fraction of the quantity then add it to the original quantity.

To find a fractional decrease, first find the fraction of the quantity then subtract it from the original quantity.

Example 1: Increase £45 by $\frac{4}{9}$

Step 1: Find $\frac{4}{9}$ of £45:

45 ÷ 9 x 4 = £20

Step 2: This is a fractional increase question, so we add to the original quantity:

45 + 20 = £65

Example 2: Due to a bad summer, a farmer forecasts that her potato crop will be $\frac{2}{5}$ lower than the previous year. She harvested 55 tonnes last year. What will it be this year?

Step 1: Find $\frac{2}{5}$ of 55 tonnes:

55 ÷ 5 x 2 = 22 tonnes

Step 2: The potato crop will be $\frac{2}{5}$ LOWER so it is a fractional decrease, so we subtract from the original quantity:

55 – 22 = 33 tonnes

Finding a Fraction of a Quantity

To find a fraction of a quantity we use the rule:

Rule: "Divide by the bottom, times by the top"

So, we divide the quantity by the denominator, then multiply the answer by the numerator.

Example 1: Calculate $\frac{3}{4}$ of 20

20 ÷ 4 x 3

= 5 x 3

= 15

Example 2: Calculate $\frac{5}{7}$ of 17.5

(calculator question)

17.5 ÷ 7 x 5

(Type straight into calculator)

= 12.5

Writing One Number as a Fraction of Another

Example 1: Write 36 as a fraction of 54, give your answer in its simplest form.

Write as a fraction → $\frac{36}{54} = \frac{2}{3}$ ← Simplify as much as possible

Example 2: In a school of 280 pupils, 120 are boys. In its simplest form, what fraction of the pupils at the school are girls? 280 – 120 = 160 girls

Number of girls → 160; Total number of pupils → 280

$\frac{160}{280} = \frac{4}{7}$ ← Simplify as much as possible

Mathematics

Higher

Unit 5

Multiplying Fractions

To multiply fractions:

- Multiply both numerators;
- multiply both denominators;
- simplify the answer if possible, or cancel down within the question before multiplying

Example 1: $\frac{2}{5} \times \frac{5}{8}$ (simplifying the answer)

Numerators multiplied, 2 x 5 = 10

Denominators multiplied, 5 x 8 = 40

$$\frac{2}{5} \times \frac{5}{8} = \frac{10}{40} = \frac{1}{4}$$

÷ 10 (numerator and denominator)

Example 2: $\frac{2}{5} \times \frac{5}{8}$ (simplifying within the question)

Cancel down any numerator and denominator. This means cancel down 2 and 8 by dividing by 2. Then cancel down both 5s by dividing by 5.

$$\frac{\cancel{2}^{1}}{\cancel{5}_{1}} \times \frac{\cancel{5}^{1}}{\cancel{8}_{4}} = \frac{1}{4}$$

Example 3: $\frac{4}{7} \times 3$ (multiplying a fraction by a whole number)

Write the 3 as a fraction by putting it over 1. Then multiply as above.

$$\frac{4}{7} \times \frac{3}{1} = \frac{12}{7} = 1\frac{5}{7}$$

Dividing Fractions

To divide fractions:

- Keep the first fraction the same;
- change the sign from a divide to a multiply;
- flip the second fraction upside down
- continue as you would for multiplying fractions

Example 1: $\frac{3}{4} \div \frac{5}{16}$ (dividing a fraction by a fraction)

$$\frac{3}{4} \div \frac{5}{16} = \frac{3}{4} \times \frac{16}{5} = \frac{48}{20} = \frac{12}{5} = 2\frac{2}{5}$$

÷ 4 (numerator and denominator)

Example 2: $\frac{9}{15} \div 3$ (dividing a fraction by a whole number)

$$\frac{9}{15} \div \frac{3}{1} = \frac{\cancel{9}^{3}}{15} \times \frac{1}{\cancel{3}_{1}} = \frac{3}{15} = \frac{1}{5}$$

Write the 3 as a fraction by putting it over 1. Then continue as above.

Example 3: $2\frac{2}{3} \div \frac{3}{5}$ (dividing fractions involving mixed numbers)

Write the mixed number as an improper fraction first.

$$\frac{8}{3} \div \frac{3}{5} = \frac{8}{3} \times \frac{5}{3} = \frac{40}{9} = 4\frac{4}{9}$$

Mathematics

Higher

Unit 5

Adding and Subtracting Fractions

We can only add or subtract fractions with the same denominators.

Example: $\frac{2}{8} + \frac{3}{8} = \frac{5}{8}$ OR $\frac{5}{9} - \frac{1}{9} = \frac{4}{9}$

When the denominators are different, we must change each fraction to have the same denominator.

Example 1: $\frac{6}{15} + \frac{2}{10}$

The lowest common multiple of the denominators, 15 and 10, is 30.
This means we want both fractions to have a denominator of 30.

$$\frac{6}{15} = \frac{12}{30} \quad (\times 2) \qquad \frac{2}{10} = \frac{6}{30} \quad (\times 3)$$

We now add $\frac{12}{30}$ and $\frac{6}{30}$ to obtain our answer.

$$\frac{12}{30} + \frac{6}{30} = \frac{18}{30} = \frac{3}{5}$$

Simplify the answer if possible

Example 2: $3\frac{4}{9} - \frac{5}{6}$

Write the mixed number as an improper fraction first.

$$3\frac{4}{9} - \frac{5}{6} = \frac{31}{9} - \frac{5}{6}$$

The lowest common multiple of the denominators, 9 and 6, is 18.
This means we want both fractions to have a denominator of 18.

$$\frac{31}{9} = \frac{62}{18} \quad (\times 2) \qquad \frac{5}{6} = \frac{15}{18} \quad (\times 3)$$

We now subtract $\frac{15}{18}$ from $\frac{62}{18}$ to obtain our answer.

$$\frac{62}{18} - \frac{15}{18} = \frac{47}{18} = 2\frac{11}{18}$$

Change back into a mixed number

Mathematics

Higher

Unit 5

Example 3: $8 + 2\frac{3}{7}$

Write the mixed number as an improper fraction first, write the 8 as a fraction by putting it over 1.

$$8 + 2\frac{3}{7} = \frac{8}{1} + \frac{17}{7}$$

The lowest common multiple of the denominators, 1 and 7, is 7. This means we want both fractions to have a denominator of 7.

$$\frac{8}{1} \overset{\times 8}{=} \frac{56}{7} \qquad \frac{17}{7}$$

Already a denominator of 7, so leave the fraction as it is.

We now add $\frac{56}{7}$ and $\frac{17}{7}$ to obtain our answer.

Change back into a mixed number

$$\frac{56}{7} + \frac{17}{7} = \frac{73}{7} = 10\frac{3}{7}$$

Problem Solving

Example 1: Idris comes from a very large family. He has many relatives, all of whom live in Canada, Japan or Wales.

$\frac{1}{5}$ of his relatives live in Canada, $\frac{3}{8}$ of his relatives live in Japan.

All 34 of his other relatives live in Wales.

How many relatives does Idris have altogether?

Step 1: Find equivalent fractions of $\frac{1}{5}$ and $\frac{3}{8}$ so they have the same denominator.

$$\frac{1}{5} = \frac{8}{40} \qquad \frac{3}{8} = \frac{15}{40}$$

Step 2: Work out what fraction of relatives live in Wales.

$$\frac{8}{40} + \frac{15}{40} = \frac{23}{40} \qquad \frac{40}{40} - \frac{23}{40} = \frac{17}{40}$$

So, 34 relatives is equivalent to $\frac{17}{40}$

Step 3: Work out the total number of relatives.

$\frac{17}{40}$ of a number = 34 → $\frac{17}{40}$ of $x = 34$

$$x = 34 \times \frac{40}{17} = 80 \text{ relatives}$$

Ratio and Proportion

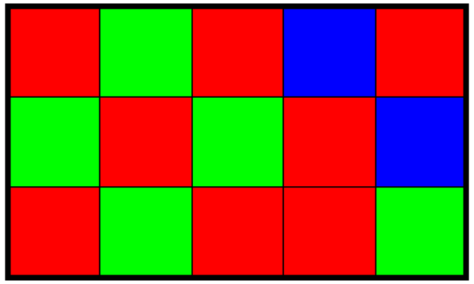
Mathematics

Higher

Unit 6

Writing Ratios

Ratios require the use of a colon **:**



The ratio of red squares to green squares is:

8 : 5

Because for every 8 red squares, there are 5 green:

The ratio of green squares to red squares is:

5 : 8

The ratio of blue squares to red squares is:

2 : 8

Simplifying Ratios

Method Just like with fractions, whatever you multiply/divide one side by, make sure you do the exact same to the other side. Keep dividing until each side has no common factors

Example 1: Simplify 14 : 21

We are looking for factors common to both sides, let's try 7.

Divide both sides by 7

÷7 (14 : 21) ÷7

2 : 3

Check: Are there are any other common factors to simplify it further? No, we have simplified it as far as possible.

Example 2: Simplify 60 : 45

We are looking for factors common to both sides, let's try 5.

Divide both sides by 5

÷5 (60 : 45) ÷5

12 : 9

Check: Are there are any other common factors to simplify it further? Yes, 3 is a common factor to both sides.

Divide both sides by 3

÷3 (12 : 9) ÷3

4 : 3

Check: Are there are any other common factors to simplify it further? No, we have simplified it as far as possible.

Note: For example 2 we could have divided both sides by 15 to start, which would have given us our answer of 4 : 3 in one step. It does not matter which way you choose, just make sure you simplify as much as possible.

Mathematics

Higher

Unit 6

In the Form "1 to n" or "n to 1"

Sometimes you may be asked to express a ratio in the form 1 : n or n : 1.

This just involves simplifying ratios. (Remember **n** is just a number).

Example 1: Express 8 : 14 in the form 1 : n

The question is asking you to change 8 : 14 into the form 1 : n, where n is just a number for you to find.

Remember, when you simplify ratios, whatever you multiply/divide one side by, do the exact same to the other side.

We can change the 8 into a 1 by dividing it by itself.

8 : 14

$\div 8$ (both sides) $\div 8$

1 : ?

Dividing the 14 by 8 gives us our final answer

1 : 1.75

Example 2: Express 0.3 : 0.15 in the form n : 1

We need to change 0.3 : 0.15 into n : 1.

We can change the 0.15 into a 1 by dividing it by itself.

0.3 : 0.15

$\div 0.15$ (both sides) $\div 0.15$

? : 1

Dividing the 0.3 by 0.15 gives us our final answer

2 : 1

Classic Ratio Question

Remember: Whatever you multiply/divide one side by, do the same to the other.

Example:

Emma is making a cake. On the packet it says that the ingredients must be mixed in the following ratios:

Flour (g)	:	Butter (g)	:	Eggs	:	Sugar (g)
400	:	220	:	3	:	25

(a) If my Emma has 1000g of flour, how much butter does she need?

(b) If she has 2 eggs, how much sugar does she need?

Always set these sort of questions out the same way – write the original ratios on the top, write the new amount you know on the bottom, and ask yourself: "what do I need to do to get from my original amount to my new amount?"

(a) This is what we've got:

flour : butter

400 : 220

$\times 2.5$ (both sides) $\times 2.5$

1000 : ?

$1000 \div 400 = 2.5$

How do I get from 400 to 1000? I multiply by 2.5, do the same to the butter.

220 x 2.5 = 550g

(b) This is what we've got:

eggs : sugar

3 : 25

$\times \frac{2}{3}$ (both sides) $\times \frac{2}{3}$

2 : ?

$2 \div 3 = \frac{2}{3}$

How do I get from 3 to 2? I multiply by $\frac{2}{3}$, do the same to the sugar.

$$25 \times \frac{2}{3} = 16\frac{2}{3}\text{g}$$

Mathematics

Higher

Unit 6

Sharing in a Given Ratio

Method for Sharing Ratios

Step 1: Add up the total number of parts you are sharing between

Step 2: Work out how much one part gets

Step 3: Use this to work out how much everybody gets.

Example 1:

24 chocolates are to be shared between Mary and Jacob in the ratio 5 : 3. Work out how many chocolates each person gets.

Step 1: Mary gets 5 parts and Jacob gets 3 parts, so in total there are 8 parts.

Step 2: There are 24 pieces of chocolate all together, so one part is worth

$24 \div 8 = 3$ pieces

Step 3: Mary has 5 parts:

$5 \times 3 = 15$ pieces

Jacob has 3 parts:

$3 \times 3 = 9$ pieces

15 + 9 = 24

Example 2:

Share £845 in the ratio 8 : 3 : 2

Step 1: In total there are 13 parts (8 + 3 + 2)

Step 2: We have £845 to share, so one part is worth

$845 \div 13 = £65$

Step 3:

8 parts	$8 \times 65 = £520$
3 parts	$3 \times 65 = £195$
2 parts	$2 \times 65 = £130$

Check: 520 + 195 + 130 = £845

In this example you do not know the total amount.

Example 3:

Tom and Lisa share money in the ratio 8:3. Tom has £40, how much does Lisa have?

Tom gets 8 parts which is worth £40.

One part is worth

$40 \div 8 = £5$

Lisa has 3 parts:

$3 \times £5 = £15$

You may even be asked how much money was there altogether. In this example £40 + £15 = £55

Mathematics

Higher

Unit 6

Ratio in Scale Drawings or Maps

For ratio problems involving scale drawings or maps, write the ratio as 'map: real life' and be careful with units. You will probably be required to convert between units.

Remember: Whatever you multiply/divide one side by, do the same to the other.

Example:

Kate and Ben planned a cycle ride using a 1:25 000 scale map. The route they planned measured approximately 80cm on the map.

a) Calculate how far they planned to cycle. Give your answer in km.
b) After the ride, Kate's watch showed they had travelled 24km. What was this measurement on the map in cm?

Always write the original ratios on the top with units

a) map : real life

1cm : 25 000 cm

80cm : ?

How do I get from 1cm to 80cm? Multiply by 80 so do the same to 25 000cm.

25 000 x 80 = 2 000 000cm

÷ by 100 to get into metres

2 000 000cm = 20 000m

÷ by 1000 to get into kilometres

20 000m = 20km

80cm on the map is equivalent to 20km in real life

b) map : real life

1cm : 25 000 cm

? : 24km

? : 2 400 000cm

Covert km into cm first (x 1000 and then x 100)

How do we get from 25 000 to 2 400 000?

2 400 000 ÷ 25 000 = 96, so multiply by 96

1 x 96 = 96cm

24km in real life is equivalent to 96cm on the map

Mathematics

Higher

Unit 6

Problems Involving Ratio, Direct Proportion and Inverse Proportion

The unitary method is used to solve simple problems involving ratio and direct proportion e.g. recipes, best buys/value for money. This involves calculating the **value** for a single item.

When working with best value in monetary terms we use: $Price\ per\ unit = \frac{price}{quantity}$

Example 1:

Box A has 8 fish fingers costing £1.40.
Box B has 20 fish fingers costing £ 3.40.
Which box is the better value?

$A = \frac{£1.40}{8} = £0.175$

$B = \frac{£3.40}{20} = £0.17$

Therefore, Box B is better value as each fish finger costs less.

In recipe terms we use: $Weight\ per\ unit = \frac{weight}{quantity}$

Example 2:

The recipe shows the ingredients needed to make 10 Flapjacks. How much of each will be needed to make 25 flapjacks?

Ingredients for 10 Flapjacks

80 g rolled oats

60 g butter

30 m*l* golden syrup

36 g light brown sugar

Oats: 80 ÷ 10 = 8
8 × 25 = **200g**

Syrup: 30 ÷ 10 = 3
3 × 25 = **75g**

Butter: 60 ÷ 10 = 6
6 × 25 = **150g**

Sugar: 36 ÷ 10 = 3.6
3.6 × 25 = **90g**

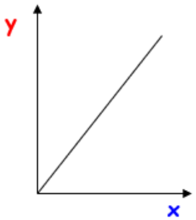
What does Proportion mean? If two variables are proportional to each other, it just means that they are related to each other in a specific way.

Two Types of Proportion - You will need to recognise and interpret graphs that illustrate direct and inverse proportion.

(a) Direct Proportion

Both variables increase or decrease together

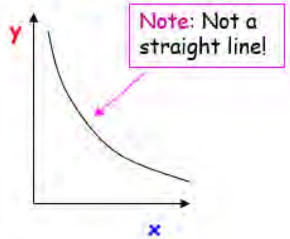
Graph



(b) Inverse Proportion

As one variable goes up, the other goes down

Graph



Examples:

Direct proportion:

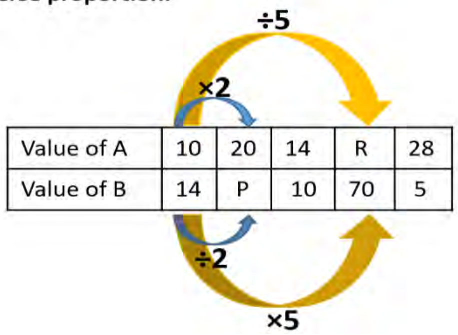
Value of A	32	P	56	20	72
Value of B	20	30	35	R	45

Ratio constant: $20 \div 32 = \frac{5}{8}$

From A to B we will multiply by $\frac{5}{8}$.
From B to A we will divide by $\frac{5}{8}$.

$P = 30 \div \frac{5}{8} = 48$ $R = 20 \times \frac{5}{8} = 12.5$

Inverse proportion:



Value of A	10	20	14	R	28
Value of B	14	P	10	70	5

So, if the value of A is doubled then the value of B is halved.

Therefore, P = 7 and R = 2

Mathematics

Higher

Unit 7

Units

Metric units of **length**:

mm, cm, m, km

Metric units of **weight**:

g, kg, tonne

Metric units of **capacity**:

ml, cl, litre

1ml = $1cm^3$

1 litre = $1000cm^3$

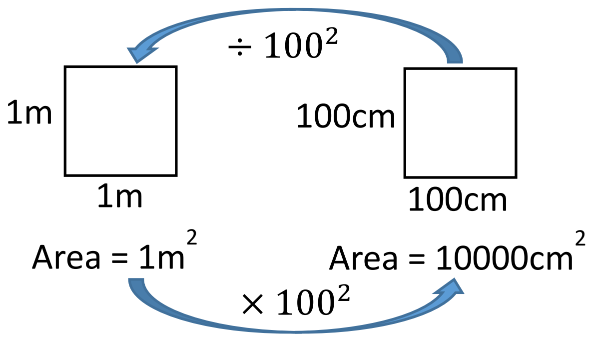
$1m^2 = 10000cm^2$

Converting length:

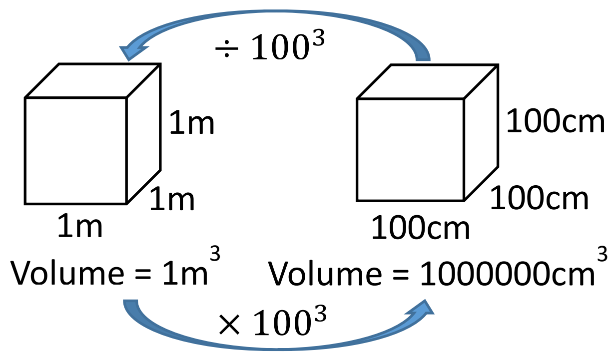
mm → cm → m → km: $\div 10$, $\div 100$, $\div 1000$

km → m → cm → mm: $\times 1000$, $\times 100$, $\times 10$

Converting area:



Converting volume:



Converting capacity:

ml → l → cl: $\div 1000$, $\div 100$

cl → l → ml: $\times 100$, $\times 1000$

Converting mass:

g → kg → tonne: $\div 1000$, $\div 1000$

tonne → kg → g: $\times 1000$, $\times 1000$

Metric- Imperial approximations

1 mile ≈ 1.6km — km → miles: $\div 1.6$; miles → km: $\times 1.6$

1kg≈ 2.2pounds — pounds → kg: $\div 2.2$; kg → pounds: $\times 2.2$

1 litre ≈ 1.75 pints — pints → litre: $\div 1.75$; litre → pints: $\times 1.75$

Mathematics

Higher

Unit 7

Example 1:

A jug holds one and a half litres of water when full.
A tank has dimensions 25 cm by 24 cm by 20 cm.

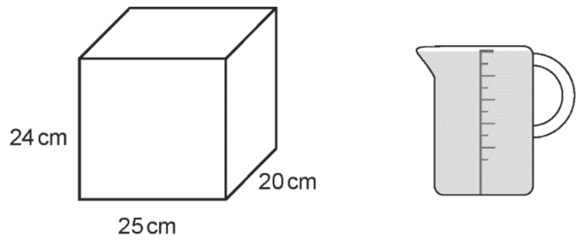


Diagram not drawn to scale

How many full jugs of water will it take to fill the tank?

Volume of tank $= 24 \times 25 \times 20$
$= 12000cm^3$

Jug holds 1.5 litres (1 litre = $1000cm^3$)
$1.5 \times 1000 = 1500cm^3$

Number of full jugs needed to fill tank:
$12000 \div 1500 = 8$

8 full jugs are needed to fill the tank.

Example 2:

How many metres are there in 5·07 kilometres? To change from km to $m \times 1000$

$5.07 \times 1000 = 5070m$

Example 3:

(a) Change 600 mm to cm. To change from mm to cm $\div 10$

$600 \div 10 = 60cm$

(b) Change 2800 mm to m.

Change from mm to cm first $2800 \div 10 = 280cm$

Then change from cm to m $280 \div 100 = 2.8m$

(c) The hotel was 3 km from the port.

(i) How far is this in metres? To change from km to m $\times 1000$

$3 \times 1000 = 3000m$

(ii) How far is this in miles? Give your answer correct to the nearest mile.

To change from km to $miles$ $\div 1.6$

$3 \div 1.6 = 1.875\ miles$

$= 2\ miles$ (to nearest mile)

Mathematics

Higher

Unit 8

Percentages

A percentage is just a fraction whose denominator (bottom) is 100.
So, if we say "32%", what we mean is $\frac{32}{100}$ or 32 out of 100.

Percentage of an Amount – Non-Calculator

Method We can calculate all percentages by first calculating some of these:

Example: You have £320. Find (a) 15%, (b) 63%, (c) 17.5%

Start by writing down the percentages that you know which might help:

To find 10% → Divide by 10 → 320 ÷ 10 = 32 → 10% = £32
To find 1% → Divide by 100 → 320 ÷ 100 = 3.2 → 1% = £3.20
To find 50% → Divide by 2 → 320 ÷ 2 = 160 → 50% = £160
To find 20% → Double 10% → 32 x 2 = 64 → 20% = £64
To find 5% → Half 10% → 32 ÷ 2 = 16 → 5% = £16
To find 2.5% → Half 5% → 16 ÷ 2 = 8 → 2.5% = £8

You can build up your answers with a bit of simple addition.

(a) 15%

15% = 10% + 5%
= £32 + £16
= £48

(b) 63%

63% = 50% + 10% + 1% + 1% + 1%
= £160 + £32 + £3.20 + £3.20 + £3.20
= £201.60

(c) 17.5%

17.5% = 10% + 5% + 2.5%
= £32 + £16 + £8
= £56

Percentage of an Amount – Calculator

Finding a percentage of an amount using a calculator can be done in one easy step.

Example:

Find 23% of 135g

Step 1: Type into the calculator

23 ÷ 100 x 135 =

Make sure you write the workings down as well as the answer.

23 ÷ 100 x 135 = 31.05g

Percentage of an Amount – Using Multipliers

We can calculate a percentage of an amount using multipliers.

- First find the multiplier by dividing the percentage by 100.
- Then multiply the multiplier by the amount given.

Example 1:

Find 4% of £22.45

Step 1: Find the multiplier

4 ÷ 100 = 0.04

Step 2: Multiply the multiplier by the amount given

0.04 x 22.45 = £0.90 (2dp)

Note: The answer has been rounded to 2dp as we are dealing with money

Example 2:

Find 31.8% of 88

Step 1: Find the multiplier

31.8 ÷ 100 = 0.318

Step 2: Multiply the multiplier by the amount given

0.318 x 88 = 27.984

Mathematics

Higher

Unit 8

Percentage Increase

If we increase an amount it means it will get bigger.
So to increase an amount by 10%, we find 10% of the amount and add it on.

Example 1 – Non-Calculator:

Increase £250 by 15%

Step 1: Find 15% of 250

10% = 25

5% = 12.5

15% = 25 + 12.5

= £37.50

Step 2: Increase means to add on

250 + 37.50 = £287.50

Example 2 – Calculator:

Increase £250 by 15%

Step 1: Type into the calculator

15 ÷ 100 x 135 =

Make sure you write the workings down as well as the answer.

15 ÷ 100 x 135 = £37.50

Step 2: Increase means to add on

250 + 37.50 = £287.50

Percentage Increase Using Multipliers

The original amount is 100%, to increase by 23% means to add 23% onto it, so we want 123%.
Find the multiplier by dividing the percentage by 100.
Then multiply the multiplier by the amount given.

Example 3 – Using Multipliers:

Increase £250 by 15%

Step 1: Find the multiplier

100 + 15 = 115

115 ÷ 100 = 1.15

Step 2: Multiply the multiplier by the amount given

1.15 x 250 = £287.50

Each different method gives the same answer.

Mathematics

Higher

Unit 8

Percentage Decrease

If we decrease an amount it means it will get smaller.
So to decrease an amount by 25%, we find 25% of the amount and take it away.

Example 1 – Non-Calculator:

Decrease 350g by 21%

Step 1: Find 21% of 350

10% = 35

1% = 3.5

21% = 35 + 35 + 3.5

= 73.5g

Step 2: Decrease means to take away

350 - 73.5 = 276.5g

Example 2 – Calculator:

Decrease 350g by 21%

Step 1: Type into the calculator

21 ÷ 100 x 350 =

Make sure you write the workings down as well as the answer.

21 ÷ 100 x 350 = 73.5g

Step 2: Decrease means to add on

350 - 73.5 = 276.5g

Percentage Decrease Using Multipliers

The original amount is 100%, to decrease by 23% means to take 23% away from it, so we want 77%.
Find the multiplier by dividing the percentage by 100.
Then multiply the multiplier by the amount given.

Example 3 – Using Multipliers:

Decrease 350g by 21%

Step 1: Find the multiplier

100 – 21 = 79

79 ÷ 100 = 0.79

Step 2: Multiply the multiplier by the amount given

0.79 x 350 = 276.5g

Each different method gives the same answer.

Mathematics

Higher

Unit 8

Percentage Change

You use this when you want to find out by what percentage an amount has gone up or down by.

We use the formula: $\text{Percentage Change} = \frac{\text{New value} - \text{Old value}}{\text{Old Value}} \times 100$

Example 1:

A pupil's marks in their maths test went from 34 to 46. What percentage increase is this?

Using the formula:

$\text{Percentage Change} = \frac{\text{New value} - \text{Old value}}{\text{Old Value}} \times 100$

New Value = 46, Old Value = 34

$\text{Percentage Change} = \frac{46 - 34}{34} \times 100$

$= \frac{12}{34} \times 100$

= 35.3% (1dp)

Example 2:

Scientific calculators have been reduced in price from £4.99 to £3.50. What percentage decrease is this?

Using the formula:

$\text{Percentage Change} = \frac{\text{New value} - \text{Old value}}{\text{Old Value}} \times 100$

New Value = 3.50, Old Value = 4.99

$\text{Percentage Change} = \frac{3.50 - 4.99}{4.99} \times 100$

$= \frac{-1.49}{4.99} \times 100$

= - 29.9% (1dp)

(The minus sign just means it is a decrease)

One Number as a Percentage of Another

Example 1 - Non-Calculator:

Write 19 as a percentage of 25?

Step 1: Write as a fraction and multiply it by 100

$\frac{19}{25} \times 100$

Step 2: Multiply (look back at Unit 5 to recall how to multiply a fraction by a whole number)

$\frac{19}{\cancel{25}_1} \times \frac{\cancel{100}^4}{1} = 76\%$

(19 is 76% of 25)

Example 2 - Calculator:

Write 256 as a percentage of 780?

Step 1: Type into the calculator

256 ÷ 780 x 100 =

Make sure you write the workings down as well as the answer.

256 ÷ 780 x 100 = 32.82% (2 dp)

(256 is 32.82% of 780)

Mathematics

Higher

Unit 8

Compound Interest

Example – Non-Calculator:

The bank pays me a compound interest rate of 5% on my balance each year. At the start I have £800 in there. How much do I have after 3 years?

Common Misconception:

Working out what 5% of £800 is, and then multiply this by 3.

This is not correct, because you don't just earn 5% on the £800, you earn it on however much is in your bank at the end of each year, which is always growing.

What we are looking for is 5% of £800 and then adding it on for the first year. This then gives us a new balance to find 5% on for the second year and so on.

This is how we set it up:

End of Year 1 5% of 800 = 40

800 + 40 = £840 (At the end of year 1 there is £840 in the bank, this is the new balance to use in year 2)

End of Year 2 5% of 840 = 42

840 + 42 = £882 (At the end of year 2 there is £882 in the bank, this is the new balance to use in year 3)

End of Year 3 5% of 882 = 44.1

882 + 44.1 = £926.10

At the end of 3 years there is £926.10 in the bank.

If we were asked to work out how much interest was added, we just need to subtract the original balance from the end balance. 926.10 – 800 = £126.10 interest added.

Compound Interest

Example – Calculator:

£5200 is invested at 2% compound interest per annum.

Calculate the total amount in the bank after 25 years.

Calculate how much interest was added.

We are going to use the formula:

$$P \times (1 \pm r/100)^n$$

P = amount in bank

r = percentage

n = number of years

From the question, P = 5200, r = 2, and n = 25

Interest is added

$$5200 \times \left(1 + \frac{2}{100}\right)^{25} = 8531.15117...$$

Total amount in bank after 25 years = £8531.15 (2 dp)

Interest added = 8531.15117... – 5200 = £3331.15 (2 dp)

Note: $\left(1 + \frac{2}{100}\right)$ is our multiplier

100% + 2% = 102%

102 ÷ 100 = 1.02

Mathematics

Higher

Unit 8

Depreciation

Example – Calculator:

James bought a car for £3500.

The car depreciates in value by 24% each year.

Find the value of the car after 4 years.

We are going to use the formula:

$$P \times (1 \pm r/100)^n$$

P = original value of car

n = number of years

r = percentage

From the question, P = 3500, r = 24, and n = 4

Depreciate means to reduce in value

$$3500 \times \left(1 - \frac{24}{100}\right)^4 = 1167.67616$$

The value of the car after 4 years is £1,167.68 (2 dp)

Note: $\left(1 - \frac{24}{100}\right)$ is our multiplier

100% - 24% = 76%

76 ÷ 100 = 0.76

Reverse Percentages

These are questions where something has been increased or decreased by a percentage and you need to work out the original amount.

The key to spotting reverse percentage questions are words such as "used to", "old" and "before" - words that suggest you need to work out something that happened in the past.

Example 1:

Sam's wages were increased by 15% to £1725

What was Sam's original wage?

Step 1: The original amount is 100%

It was increased by 15% to 115%

So 1725 = 115%

Step 2: Work out 1%

Divide both sides by 115 to find 1%

15 = 1%

Multiply by 100 to get the original 100%

1500 = 100%

Sam's original wage was £1500.

Example 2 – Using Multipliers:

The value of a car decreased by 23% to £654.50

What was the car worth before the decrease?

$$w \times 0.77 = 654.50$$

- w: The old value of the car
- 0.77: The old value was decreased by 23%
- 654.50: To give the new value

Divide both sides by 0.77

$w = 654.50 \div 0.77$

$w = £850$

The original value of the car was £850.

Mathematics

Higher

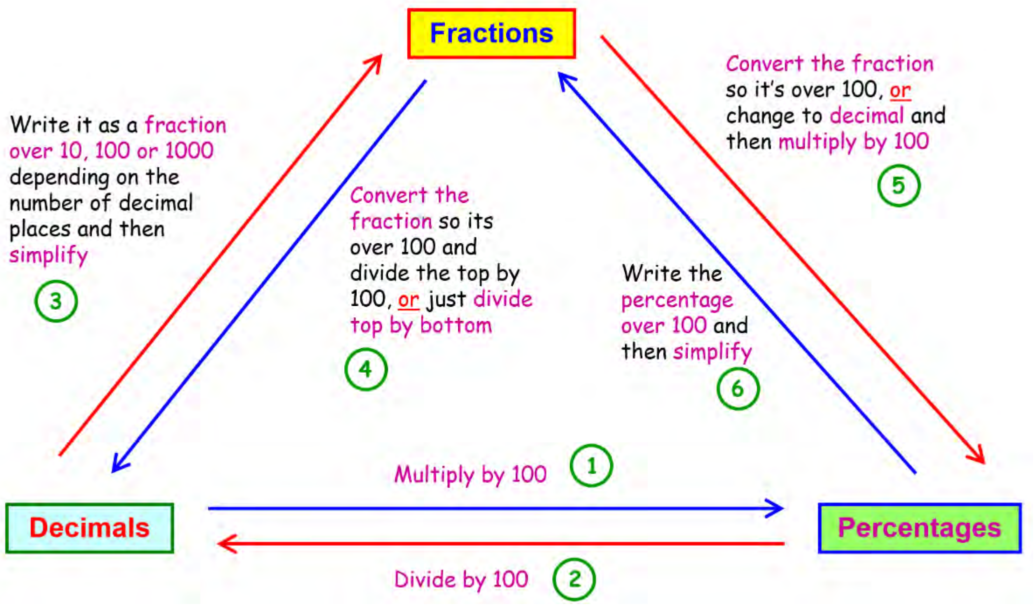
Unit 9

Fractions, Decimals and Percentages

Fractions, Decimals and Percentages are all closely related to each other, and you need to be comfortable changing between each of them.

You can use this diagram to help you.

Follow the arrows depending on what you need to change and follow the numbers for the examples.



Examples:

1 What is 0.364 as a percentage?

Just multiply by 100 and be careful with the decimal point!

0.364×100

$= 36.4\%$

2 Convert 8.3% into a decimal

Just divide by 100 and again be careful with the decimal point!

$8.3 \div 100$

$= 0.083$

3 Write 0.16 as a fraction

There are 2 decimal places, so write it over 100

$\frac{16}{100}$

Now carefully simplify

$\frac{16}{100} = \frac{8}{50} = \frac{4}{25}$

4 Write $\frac{13}{20}$ as a decimal

We need to change the bottom of the fraction to 100, remembering to do the same to the top

$\frac{13}{20} = \frac{65}{100}$ (× 5 top and bottom)

Divide the top of your fraction by 100 and you have your answer!

$= 0.65$

5 Write $\frac{5}{8}$ as a percentage

It's not easy to change this fraction over 100, so we must divide 5 by 8

$5 \div 8$

Use any method, but I do this: $= 8\overline{)5.000}$ giving 0.625

0.625 is the answer as a decimal, so we must multiply by 100

0.625×100

$= 62.5\%$

6 What is 12.5% as a fraction?

Start by writing the percentage over 100

$\frac{12.5}{100}$

We need to simplify, but the decimal point makes it hard. So why not multiply top and bottom by 2!

$\times 2$ $\frac{25}{200}$

Now we can simplify as normal to get the answer:

$\frac{25}{200} = \frac{5}{40} = \frac{1}{8}$

Mathematics

Higher

Unit 9

Ordering Fractions, Decimals and Percentages

To order a mix of fractions, decimals, and percentages you need to first convert all the numbers to the same form, either fractions, decimals, or percentages.

Note: Ascending Order means smallest to largest.

Descending Order means largest to smallest.

Example:

Put the following in ascending order

56% $\frac{3}{4}$ 0.871 23% $\frac{6}{7}$

To order these, convert them all to decimals.

56%	$\frac{3}{4}$	0.871	23%	$\frac{6}{7}$
0.56	0.75	0.871	0.23	0.857...
2	3	5	1	4

Then write the correct order but as they were in the original question.

23% 56% $\frac{3}{4}$ $\frac{6}{7}$ 0.871

Here are some equivalent fractions, decimals, and percentages you should know.

F	D	P
$\frac{1}{100}$	0.01	1%
$\frac{1}{10}$	0.1	10%
$\frac{1}{5}$	0.2	20%
$\frac{1}{4}$	0.25	25%
$\frac{1}{2}$	0.5	50%
$\frac{3}{4}$	0.75	75%
$\frac{1}{3}$	$0.\dot{3}$	$33.\dot{3}\%$
$\frac{2}{3}$	$0.\dot{6}$	$66.\dot{6}\%$

Recurring Decimals

Some decimals **terminate**, which means the decimals do not recur, they just stop. For example, 0.75.

A **recurring decimal** exists when decimal numbers repeat forever.

Convert $\frac{8}{11}$ into a decimal using your calculator. A calculator displays this as $0.\dot{7}\dot{2}$ or 0.7272727272......

The digits 2 and 7 repeat infinitely. This is an example of a **recurring decimal**.

We can show this by writing dots above the 7 and the 2 (the numbers that recur).

If you had to convert into a recurring decimal without the calculator, you would need to use the bus shelter method

Write $\frac{5}{6}$ as a decimal

$$6\,\overline{)5.0000} = 0.8333$$

So, $\frac{5}{6} = 0.8\dot{3}$

Here are some more examples of recurring decimals:

$\frac{4}{9} = 0.\dot{4}$ This decimal is made up of an infinite number of repeating 4s.

$\frac{5}{6} = 0.8\dot{3}$ This decimal starts with an 8 and is followed by an infinite number of repeating 3s.

$\frac{2}{7} = 0.\dot{2}8571\dot{4}$ In this decimal, the six digits 285714 repeat an infinite number of times in the same order.

$\frac{9}{22} = 0.4\dot{0}\dot{9}$ This decimal starts with a 4. The two digits 09 then repeat an infinite number of times.

Mathematics

Higher

Unit 10

Everyday Maths

Income tax

Key Words

Gross income: Money earned (salaries, bonuses etc.)
Taxable income: Money that can be taxed
Personal allowance: Money you don't have to pay tax on
Tax: A compulsory financial charge to fund government expenditures.
Per annum: Per year (annually, yearly etc.)

Method:

Step 1: Draw a diagram. Use it to calculate how much tax is payable from each tax bracket (no tax, 20%, 40%).

Step 2: Calculate the tax due in each tax bracket.

Step 3: Add the together the calculated tax values.

Step 4: Re-read the question, is it asking for annual wage

Example 1: David earns £21,000 per annum. He pays tax at 20% on any earnings over £12,500 per year. Calculate the amount of money he receives after tax each month.

Step 1:

£12500 no tax	£7500 taxed at 20%

Step 2: The tax payable at 20%

20% of £7500:

10% = £750

20% = £1500

David gets £21000 - £1500 = £19500 per annum after tax

Example 2: Claudia was given the following information:

UK Income Tax

April 2013 to April 2014

taxable income = gross income – personal allowance

- personal allowance is £9205
- basic rate of tax: 20% on the first £32 255 of taxable income
- higher rate tax: 40% is payable on all taxable income over £32 255

During the tax year 2013 to 2014, Claudia's gross income was £52 250.
Calculate the total amount of tax that Claudia should pay.
You must show all your working.

Step 1: How much income is taxable?

52250 – 9250 = £43045

£9250 No tax personal	£32255 at 20%	43045 - 32255 = £10790 £10790 at 40%

Step 2: Total tax to be paid at 20%

20% of £32225: 0.2 x 32255 = £6451

Total tax to be paid at 40%

40% of £10790: 0.4 x 10790 = £4316

Step 3: Total amount of tax payable = £6415 + £4316

= £10767

Mathematics

Higher

Unit 10

Household Bills – electricity bills, water bills, etc.

Method

Step 1: Find the number of units used.

Step 2: Calculate the cost of units used, convert to £.

Step 3: Calculate the total cost before VAT, cost of units used + service charge.

Step 4: Calculate the cost of the VAT and add the cost of VAT on the total amount.

[1 + VAT (as decimal)] x cost before VAT

Example: Ruth gets her electricity bill for the 3-month period July – September 2000.
The details are as follows:

Previous meter reading	46583
Present meter reading	49468
Charge per unit	6.65 pence per unit
Service charge	£10.56
VAT	5%

Write out the details of the cost of electricity for this period and find the total bill including VAT

Step 1: Units used = 49468 – 46583

= 2885 units

Step 2: Cost of units used = 6.65 x 2885

= 19185.25p

Convert to pounds: 19185.25 ÷ 100 = £191.8525

Step 3: Cost including service charge = £191.8525 + £10.56

= £202.4125

Step 4: Cost including 5% VAT

100% + 5% = 105%

1.05 x 202.4125 = £212.533125

= £212.53 (2 d.p.)

Mathematics

Higher

Unit 10

Distance tables

The chart below shows the road distances between some towns and cities.
The distances are given in miles.

Abergavenny			
18	**Newport**		
45	53	**Gloucester**	
50	32	36	**Bristol**

Look down the column from Abergavenny and across the row from Bristol where they meet is the answer

Wyn lives in Abergavenny and works in Bristol.

(a) Use the chart to find the road distance from Abergavenny to Bristol. 50 miles

Wyn works in Bristol for 5 days each week.
Each day, he drives to and from work using the route shown on the map.

Diagram not drawn to scale

How many miles, in total, does he travel to and from work each week?

Wyn travels from Abergavenny to Newport and then Newport to Bristol

18 + 32 = 50 miles

Return home journey = 50 miles

Wyn travels 100 miles a day

5 days a week = 5 x 100 =500

Therefore, Wyn travels 500 miles each week

One day, Wyn had to use a different route through Gloucester to get to and from work.

Alternative Route

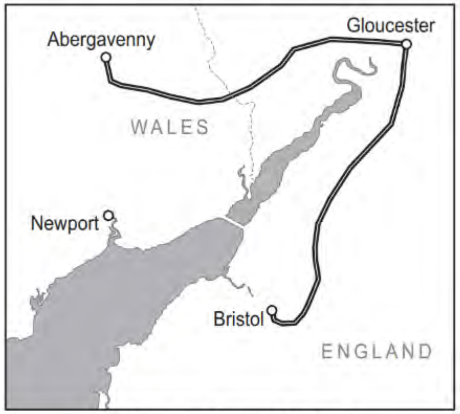


Diagram not drawn to scale

Use the chart to work out how many **extra** miles Wyn travelled that day.
You must show all your working.

Normally 100 miles a day

New route is from Abergavenny to Gloucester and then Gloucester to Bristol

45 + 36 = 81miles

Return home = 81 miles

Total distance travelled 162 miles

So, Wyn travelled 62 extra miles

Mathematics

Higher

Unit 10

Exchange rates

Method

- To convert from British pounds to a **new** currency, you multiply by the exchange rate.

e.g. The exchange rate is £1 = $2.65

So £90 in dollars would be 90 x 2.65 = $238.50.

- To convert from a **new** currency to British pounds, you divide by the exchange rate.

e.g. The exchange rate is £1 = 1.21€

So 34.50€ in British pounds would be 34.50 ÷ 1.21 = 28.51€ (to 2 d.p.)

Example 2:

Mena goes on holiday to France.
She takes 590 euros with her on holiday.

Mena only spends 40% of her euros.

When she returns from holiday, she exchanges her remaining euros for pounds.
The exchange rate is £1 = 1.18 euros.
How many pounds does Mena receive?

Mena brought 60% of her euros back: 60% of 590 euros = 0.6 x 590

=354 euros

354 euros in pounds: 354 ÷ 1.18 = £300

Example 1:

Ewan is going on holiday to India.
He has saved £450 to exchange for Indian rupees.

(a) The exchange rate on the internet last week was £1 = 99.40 rupees.
Had Ewan been going on holiday last week, how many rupees could he have bought?

450 x 99.4 = 44730 rupees

(b) Ewan exchanges his money on arrival in India.
The exchange rate is now £1 = 99.72 rupees.

The exchange bureau only has 500 rupee notes.
Ewan wants to buy as many rupees as possible with his £450 savings.

How much of his £450 will Ewan spend buying rupees?
Give your answer correct to the nearest penny.
You must show all your working.

With his money Ewan could get: 450 x 99.72 = 44874 rupees

As the bureau only has 500 rupee notes, the most rupees Ewan can have is 44500 rupees (he couldn't have 4500 rupees as he doesn't have enough pounds to exchange)

44500 rupees in pounds is: 44500 ÷ 99.72 = £446.2494……

So, to the nearest penny, Ewan will spend £446.25 buying rupees

Mathematics

Higher

Unit 10

Timetables and Time

Example 1:

The following tables are parts of train timetables between Reading and London and between London and Birmingham.

Reading	09:55	10:03	10:10	10:38	11:26
London	10:25	10:44	10:49	11:17	11:57

London	15:03	15:23	15:43	15:54	16:50
Birmingham	16:27	16:45	17:08	17:17	18:11

Andrew catches the 10:38 train from Reading to London.
How long should the journey take?

10:38 → 11:17

10:38 → 11 = 22 minutes

11 → 11:17 = 17 minutes

22 + 17 = 39 minutes

Example 2:

When it is 19:40 in Cardiff, it is 23:40 in Dubai.

19:40 → 23:40
+4hrs

(i) What time is it in Dubai when it is 13:30 in Cardiff?
Circle your answer.

15:30 10:30 09:30 (17:30) 19:30

13:30 → 17:30

+4hrs

(ii) What time is it in Cardiff when it is 02:10 in Dubai?
Circle your answer.

20:10 06:10 (22:10) 10:10 00:10

Cardiff → Dubai +4hrs

Dubai → Cardiff -4hrs

02:10 → 22:10

Mathematics

Higher

Unit 10

Best Buys

Method

- Decide how you are going to compare the offers, how many items or the mass/capacity/cost.
- Use division to get the number of items/capacities that you are going to compare.

Example 1:

Small bottle 300 ml for 66p | Medium bottle 400 ml for 92p | Large bottle 500 ml for £1.25

Roland is going to buy some orange juice for a party.
Which size bottle of orange juice offers the best value for money?
You must show your working.

Compare the capacity (100ml of each)

Small bottle: 300ml is 66p
÷3 ÷3
100ml is 22p

Medium bottle: 400ml is 92p
÷4 ÷4
100ml is 23p

Large bottle: 500ml is £1.25 = 125p
÷5 ÷5
100ml is 25p

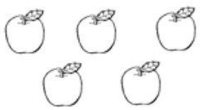
The best value for money the small bottle at 22p per 100ml.

Example 2:

1) Two shops, Kwik Stores and Bob's Fruit and Veg, both sell Pink Lady apples.

Kwik Stores

Pink Lady apples

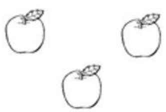


5 for £1.80

At which shop are Pink Lady apples the better value for money? Show all your working.

Bob's Fruit and Veg

Pink Lady apples



3 for £1.05

Compare the cost of 1 apple

Kwik Stores: 5 for £1.80
÷5 ÷5
1 for £0.36

Bob's Fruit and Veg: 3 for £1.05
÷3 ÷3
1 for £0.35

Bob's Fruit and Veg is cheaper by 1p per apple.

Mathematics

Higher

Unit 10

AER & APR

AER – annual equivalent rate

This gives the percentage interest earned in a savings or investment account in one year. It enables comparison of rates between different lenders and accounts which pay interest at different frequencies e.g. each month, quarter, 6 months.

APR – annual percentage rate

This measures the cost of borrowing money. The calculation includes fees charged by the lender for setting up the loan.

EAR – equivalent annual rate

Again, this measures the cost of borrowing money, though this time in the form of an overdraft.

AER as a **decimal**, is calculated using the formula:

$$\left(1+\frac{i}{n}\right)^{n}-1$$

where i is the nominal interest rate per annum as a **decimal** and n is the number of compounding periods per annum.

AER Example

A savings account is advertised as paying 4.28% nominal interest rate, with interest payments made once every 3 months. What is the AER? (Interest over 1 year)

Solution

We can use the formula to find the AER as a **decimal**:

$$\text{AER (as a decimal)} = \left(1+\frac{i}{n}\right)^{n}-1$$

where i = **nominal interest rate** per annum (4.28%) as a decimal,

n = number of compounding periods per annum (in this case n = 12 ÷ 3 = 4).

For example:

$$\left(1+\frac{0.0428}{4}\right)^{4}-1 = 0.043491$$

Then, convert your answer back into a percentage by multiplying by 100.

AER = 4.35%

Mathematics

Higher

Unit 11

Simplifying in Algebra

Key Words:

Term: This is any part of an expression or equation that involves a letter.

e.g. $4m$, $-2r$, and p are all terms

Expression: This is a collection of terms, sometimes including numbers as well, it does not have an equals sign.

e.g. $4m + 2r$ and $8z - 5p + 6q^2 - 7$ are all expressions

Equation: This is like an expression but it contains an equals sign.

e.g. $4m + 2r = 7$ and $8z + 6q^2 - 7 = a$ are all equations

Identity: This is an equation that is always true no matter what values are substituted.

Directed numbers

+ + → +

− − → +

+ − → −

− + → −

eg $a = -5$ and $b = 2$

$a^2 = a \times a = -5 \times -5 = 25$

$b + a = 2 + -5 = -3$

You can add or subtract LIKE TERMS but you cannot add or subtract DIFFERENT TERMS.

A LIKE TERM is a term that contains the exact same letter (or letters) as another term

e.g. $m + m = 2m$ $\quad 3p + 2p = 5p$ $\quad 16t^2 - 4t^2 = 12t^2$ $\quad 10pq - 7pq = 3pq$

3 lots of something, plus 2 lots of something, gives you 5 lots of something

16 lots of something, minus 4 lots of something, gives you 12 lots of something

BUT...

$m + p$ Does Not = mp $\quad 3r + 2t$ Does Not = $5rt$

Because the terms are different!

Mathematics

Higher

Unit 11

Simplifying Expressions

Adding and Subtracting

Note: To simplify an expression, draw boxes around all the LIKE TERMS and deal with each set of like terms on their own.

Example 1: Simplify $4m + 2p - m + 6p$

Remember: Draw each box around the term and the sign in front of the term.

$\boxed{4m}\ \boxed{+\ 2p}\ \boxed{-\ m}\ \boxed{+\ 6p}$

We have:

$4m - m = 3m$

$2p + 6p = 8p$

So: $4m + 2p - m + 6p = 3m + 8p$

Note: if you cannot see a sign in front of a term, then just assume it is a PLUS

Example 2: Simplify $4t^2 - 5t - 2t - 3t^2$

Remember: t and t^2 are DIFFERENT!

$\boxed{4t^2}\ \boxed{-\ 5t}\ \boxed{-\ 2t}\ \boxed{-\ 3t^2}$

We have:

$4t^2 - 3t^2 = t^2$

Note: write this instead of $1\,t^2$

$-5t - 2t = -7t$

So: $4t^2 - 5t - 2t - 3t^2 = t^2 - 7t$

Mathematics

Higher

Unit 11

Multiplying and Dividing

When multiplying and dividing with Algebra, we need to remember the following things:

- We CAN multiply different terms and like terms together
- Always multiply the numbers together first
- Put the letters in alphabetical order
- When multiplying, leave out the multiplication sign
- When dividing, watch for things cancelling out and disappearing

Example 1: Simplify $5b \times 2c \times 3a$

Step 1: Multiply the numbers together first:

$$5 \times 2 \times 3 = 30$$

Step 2: Now the letters, remembering to write them in alphabetical order and leave out the multiplication sign:

$$b \times c \times a = bca = abc$$

Step 3: Put them together, leaving out the multiplication sign:

$$5b \times 2c \times 3a = 30acb$$

Example 2: Simplify $4r \times -3p \times 3r \times q$

Step 1: Multiply the numbers together first, be careful with the negatives:

$$4 \times -3 \times 3 \times 1 = -36$$

Note: there was no number in front of the q, which means it is just a 1.

Step 2: Now the letters:

$$r \times p \times r \times q = pqrr = pqr^2$$

Remember: if you multiply something by itself, it means you are squaring it.

Step 3: Put them together:

$$4r \times -3p \times 3r \times q = -36pqr^2$$

Example 3: Simplify $\frac{5a^2b}{35ab^3}$

Step 1: Divide the numbers first:

$$5 \div 35 = \frac{5}{35} = \frac{1}{7}$$

Note: If you don't get a nice answer you can leave your answer as a fraction.

Step 2: Now the letters (this requires a bit of knowledge about indices):

The a on the bottom cancels out one a on top, but still leaves an a behind on the top

The b^3 on the bottom cancels out the b on the top, and still leaves a b^2 behind on the bottom.

Step 3: Put them together:

$$\frac{5a^2b}{35ab^3} = \frac{a}{7b^2}$$

Mathematics

Higher

Unit 11

Forming Expressions

We can form expressions for a range of problems using letters to stand for unknown values.

Example 1:

Sam's brother Tom is 3 years older than Sam. Their dad, Will, is four times as old as Sam. Form and simplify an expression for the sum of their ages.

Let us represent Sam's age by the letter x.

Sam is x years old

Tom is 3 years older than Sam

3 years older

Tom is $x+3$ years old

Will is four times as old as Sam

4 times as old

Will is $4x$ years old

The sum of their ages is represented by

Sam's age + Tom's age + Will's age

$x \quad + \quad x+3 \quad + \quad 4x$

Simplifying gives:

$x+x+4x \rightarrow 6x+3$

Example 2:

The width of a rectangle is $2x$ cm, the length of the rectangle is 5cm less than the width. Form and simplify an expression for the perimeter of the rectangle.

The perimeter of a rectangle is found by adding all the lengths of the sides together.

Width is $2x$ cm

Length is $2x-5$ cm

So, an expression for the perimeter is given by:

$2x+2x+2x-5+2x-5$ ← 2 widths and 2 lengths

Simplifying gives:

$2x+2x+2x+2x \rightarrow 8x-10 \leftarrow (-5)+(-5)$

Mathematics

Higher

Unit 11

Expanding Single Brackets

When we expand brackets, we multiply the number/term outside the bracket by each number/term inside the bracket.

$3 \times 5a = 15a$

$3(5a - 2)$

$3 \times -2 = -6$

$3(5a - 2) = 15a - 6$

Example 1: $-3(2x + 6)$

Remember, the -3 is multiplied by everything inside the bracket.

$-3 \times 2x = -6x$

$-3(2x + 6)$

$-3 \times 6 = -18$

$-3(2x + 6) = -6x - 18$

Example 2: $-10(2c - 4)$

Remember, the -10 is multiplied by everything inside the bracket.

$-10 \times 2c = -20c$

$-10(2c - 4)$

$-10 \times -4 = 40$

$-10(2c - 4) = -20c + 40$

Example 3: $6a(2a + 6)$

Remember, the $6a$ is multiplied by everything inside the bracket.

$6a \times 2a = 12a^2$

$6a(2a + 6)$

$6a \times 6 = 36a$

$6a(2a + 6) = 12a^2 + 36a$

Example 4: $-5y(4 - 2y)$

Remember, the $-5y$ is multiplied by everything inside the bracket.

$-5y \times 4 = -20y$

$-5y(4 - 2y)$

$-5y \times -2y = 10y^2$

$-5y(4 - 2y) = -20y + 10y^2$

Mathematics

Higher

Unit 11

Expanding Pairs of Single Brackets

When we expand pairs of single brackets, we separate the question into two parts, work each part out separately, then combine and simplify the answers.

$$3(5a-2) \mid + 2(2a+4)$$

$3(5a-2) = 15a - 6$

$2(2a+4) = 4a + 8$

$$15a - 6 + 4a + 8 = 19a + 2$$

Example 1: $6(x+4) + 2(x-7)$

Separate into two parts:

$$6(x+4) \mid + 2(x-7)$$

$6(x+4) = 6x + 24$

$2(x-7) = 2x - 14$

Combine and simplify:

$6x + 24 + 2x - 14 = 8x + 10$

Example 2: $5(x-2) - 3(x+1)$

Separate into two parts:

$$5(x-2) \mid - 3(x+1)$$

$5(x-2) = 5x - 10$

$-3(x+1) = -3x - 3$

Combine and simplify:

$5x - 10 - 3x - 3 = 2x - 13$

Example 3: $5(x-1) - 2(x-3)$

Be careful with the minus signs

Separate into two parts:

$$5(x-1) \mid - 2(x-3)$$

$5(x-1) = 5x - 5$

$-2(x-3) = -2x + 6$

Combine and simplify:

$5x - 5 - 2x + 6 = 3x + 1$

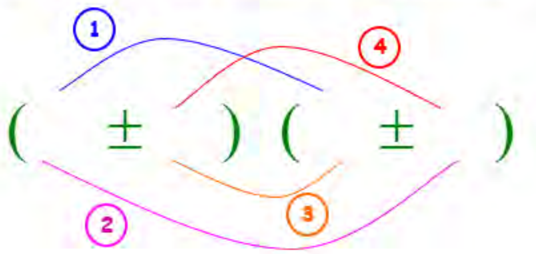
Mathematics

Higher

Unit 11

Expanding Double Brackets

To help us expand double brackets, we use the acronym FOIL (First Outer Inner Last), this tells us the order in which we need to multiply terms.



Some people call this the smiley face method.

1) **F irst** — Multiply together the first terms in each bracket - remembering to include the signs in front of them

2) **O uter** — Multiply together the terms on the outside each bracket - remembering to include the signs in front of them

3) **I nner** — Multiply together the terms on the inside each bracket - remembering to include the signs in front of them

4) **L ast** — Multiply together the last terms in each bracket - remembering to include the signs in front of them

Example 1: Expand and simplify

$(a + 6)(a + 4)$

$$(a + 6)(a + 4)$$

First $a \times a = a^2$

Outer $a \times 4 = 4a$

Inner $6 \times a = 6a$

Last $6 \times 4 = 24$

Now we write down our answers, in order, remembering if there is no sign in front of our term it's just a plus.

$$a^2 \boxed{+ 4a + 6a} + 24$$

Notice that the middle terms simplify to give

$$a^2 + 10a + 24$$

Example 2: Expand and simplify

$(p + 10)(p - 8)$

$$(p + 10)(p - 8)$$

Be careful with the negatives

First $p \times p = p^2$

Outer $p \times -8 = -8p$

Inner $10 \times p = 10p$

Last $10 \times -8 = -80$

Now we write down our answers, in order, making sure we get all the signs correct.

$$p^2 \boxed{- 8p + 10p} - 80$$

Notice that the middle terms simplify to give

$$p^2 + 2p - 80$$

Example 3: Expand and simplify

$(t - 9)(t + 2)$

$$(t - 9)(t + 2)$$

Be careful with the negatives

First $t \times t = t^2$

Outer $t \times 2 = 2t$

Inner $-9 \times t = -9t$

Last $-9 \times 2 = -18$

$$t^2 \boxed{+ 2t - 9t} - 18$$

Simplify the middle terms

$$t^2 - 7t - 18$$

Example 4: Expand and simplify

$(m - 7)^2$

$$(m - 7)(m - 7)$$

Be careful with the negatives

First $m \times m = m^2$

Outer $m \times -7 = -7m$

Inner $-7 \times m = -7m$

Last $-7 \times -7 = 49$

Writing down our answers, we get

$$m^2 \boxed{-7m - 7m} + 49$$

Simplify the middle terms

$$m^2 - 14m + 49$$

Mathematics

Higher

Unit 12

Standard Form

Standard form is a convenient way of writing out really big or really small numbers.

There is a set way of writing standard form as seen to the right.

You may see standard form written with or without brackets.

3×10^4

Must be a number between 1 and 10 but not 10

Must be multiplied by a power of 10

3×10^4
$= 30000$

3×10^{-4}
$= 0.0003$

Negative power creates a smaller number

Converting an Ordinary Number to Standard Form

Example 1: 2 3 0 0 0 0 0 0 0 0

With whole numbers like this, the decimal point is hidden at the end:

2 3 0 0 0 0 0 0 0 0.

We count how many places we need to move the decimal point until we create a number between 1 and 10

The number we want is 2.3…

2.3 0 0 0 0 0 0 0 0

We have moved the decimal point 9 places to the left, so our answer is:

$$2.3 \times 10^9$$

Example 2: 0 . 0 0 0 0 4 6 2 3

With decimals like this we can see the decimal point quite clearly.

0.0 0 0 0 4 6 2 3

We count how many places we need to move the decimal point until we create a number between 1 and 10

The number we want is 4.623…

0 0 0 0 0 4.6 2 3

We have moved the decimal point 5 places to the right, so our answer is:

$$4.623 \times 10^{-5}$$

Converting Standard Form to an Ordinary Number

Example 1: 1.02×10^6

We can see where the decimal point is, and the index number (6) tells us we must move the decimal point 6 places to the right, filling any gaps with zeros.

1.0 2 0 0 0 0

Our answer is:

1 0 2 0 0 0 0

Check: If you start with 1 0 2 0 0 0 0 and move your finger back 6 places, do you end up with…

1.02×10^6

Yes, so you've got it right.

Example 2: 7.6×10^{-5}

We can see where the decimal point is, and the index number (-5) tells us we must move the decimal point 5 places to the left, filling any gaps with zeros.

0.0 0 0 0 7.6

Our answer is:

0 . 0 0 0 0 7 6

Check: If you start with 0 . 0 0 0 0 7 6 and move your finger back 5 places, do you end up with…

7.6×10^{-5}

Yes, so you've got it right.

Mathematics

Higher

Unit 12

Tip: In the next section you may end up with answers that look like standard form but do not obey the rules. Therefore, you will need to convert to standard form.

e.g. 16×10^4 *16 is not between 1 and 10!*

Option one:

Convert to an ordinary number then convert to the accepted version of standard form

$160000 = 1.6 \times 10^5$

Option two: 16×10^4

Write as standard form

$1.6 \times 10^1 \times 10^4$

1.6×10^5

Remember your rules of indices

Standard form can be written on a scientific calculator with the 'exponent' button. The exponent button may be shown as EXP, EE or $\times 10^x$.

Multiplying and Dividing in standard form

Multiply or divide the front numbers, then add or subtract the indices using the rules of indices.

Examples:

$(3 \times 10^4) \times (2 \times 10^2)$
$3 \times 2 \times 10^4 \times 10^2$
6×10^6

$(8 \times 10^4) \div (2 \times 10^2)$
$8 \div 2 \quad 10^4 \div 10^2$
4×10^2

$(3 \times 10^4) \times (4 \times 10^2)$
$3 \times 4 \times 10^4 \times 10^2$
12×10^6
$1.2 \times 10^1 \times 10^6$
1.2×10^7

Division may be written in this form

$\frac{(3 \times 10^{-10})}{(6 \times 10^5)}$
$3 \div 6 \quad 10^{-10} \div 10^5$
0.5×10^{-15}
$5 \times 10^{-1} \times 10^{-15}$
5×10^{-16}

Adding and subtracting in standard form

When adding or subtracting in standard form we can either convert to ordinary numbers first or convert the numbers to have the same power of 10.

Example 1 $(2.3\times10^4) + (4.31\times10^5)$

$(2.3\times10^4) \rightarrow 23000$

$(4.31\times10^5) \rightarrow 431000$

We must change both numbers into normal numbers:

Now we line our digits up carefully and add...

```
  431000
+  23000
  454000
```

Usually you will then be asked to convert your answer back into Standard Form...

$454000 = 4.54 \times 10^5$

Example 2 $(1.2 \times 10^{17}) - (6.4 \times 10^{16})$ $\quad 1.2 \times 10^{17} \rightarrow 12 \times 10^{16}$

By converting the first number, both now have the same power of 10 (10^{16}). The front numbers can now be subtracted.

$(12 \times 10^{16}) - (6.4 \times 10^{16}) = 5.6 \times 10^{16}$

Make sure your final answer is in standard form.

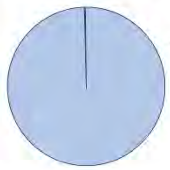
Mathematics

Higher

Unit 13

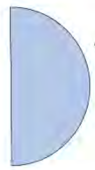
Drawing Angles and Angle Facts

What angle is …



… a full turn?

360°

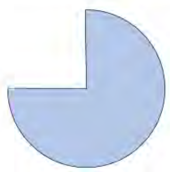


… a half turn?

180°

… a quarter turn?

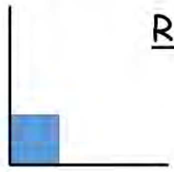
90°



… a three quarter turn?

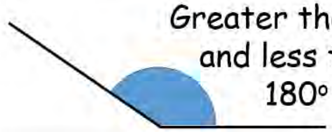
270°

What type of angle is …



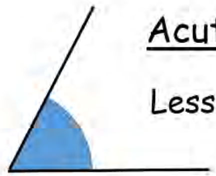
Right-Angle

Exactly 90°



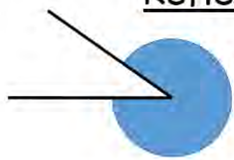
Obtuse Angle

Greater than 90° and less than 180°



Acute Angle

Less than 90°

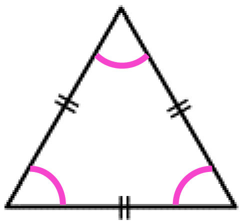


Reflex Angle

Greater than 180° and less than 360°

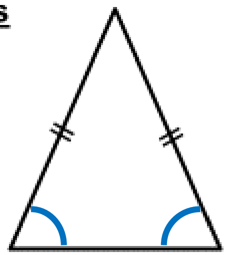
Angles in a Triangle add to 180°

Special Triangles



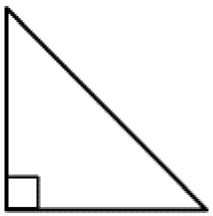
Equilateral Triangle

3 equal lengths
3 equal angles



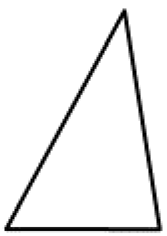
Isosceles Triangle

2 equal lengths
2 equal angles



Right-angle Triangle

1 right angle

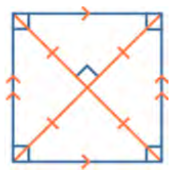


Scalene Triangle

No equal lengths
No equal angles

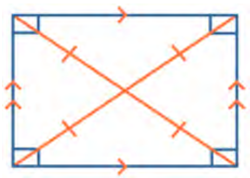
Angles in a Quadrilateral add to 360°

Special Quadrilaterals



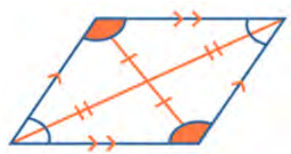
Square

- 4 equal sides
- Opposite sides are parallel
- 4 angles of 90°
- Diagonals bisect each other at 90°



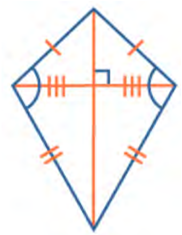
Rectangle

- Opposite sides are equal and parallel
- 4 angles of 90°
- Diagonals bisect each other



Parallelogram

- Opposite sides are equal and parallel
- Opposite angles are equal



Kite

- 2 pairs of equal sides
- One pair of opposite angles is equal

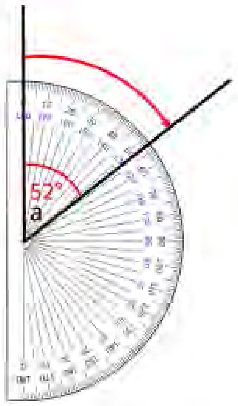
Mathematics

Higher

Unit 13

Measuring and Drawing Angles

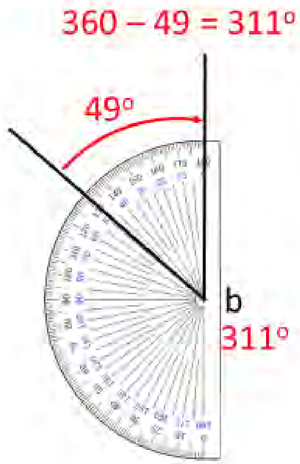
Measure angles



Example: Measure the angle marked a.

1. Place the centre point of the protractor on the corner of the angle and line up the zero line
2. Work out the direction to measure from, make sure you always read from 0°
3. Read the angle from the protractor and label

Measure reflex angles

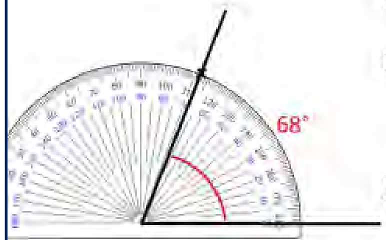


We use the fact that a full turn adds up to 360°.

Example: Measure the angle marked b.

1. Measure the smaller angle that makes up a full turn.
2. Subtract this angle from 360 to calculate the reflex angle then label.

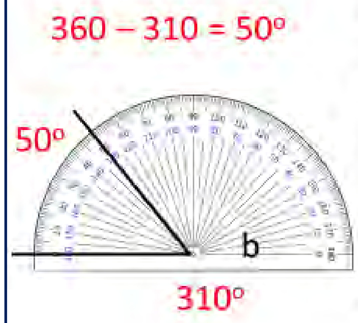
Draw angles



Example: Accurately draw an angle of 68°.

1. Draw a horizontal line
2. Place the centre point of the protractor on one end of the line. Line up the zero of the protractor with the drawn line.
3. Work out the direction, always measure from the zero, and place a mark at 68°.
4. Draw a line from the end of the line you used through the mark and label the angle. **Check using angle types!**

Draw reflex angles



We use the fact that a full turn adds up to 360°.

Example: Accurately draw an angle of 310°.

1. Subtract the angle from 360.
2. Draw an angle of this size.
3. Label the opposite angle 310°.

Mathematics

Higher

Unit 13

Angle Facts

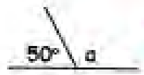
Tips for Answering Angle Questions

1. Always write down the name of each of the Angle Facts you have used to get your answer (even if there are more than one)
2. Parallel Lines are only parallel if they have the little arrows to say so!
3. If you have lots of labelled angles to find and you just don't know where to start, sometimes it's a good idea to go in alphabetical order!
4. Often there are lots of different ways of working out the answer

Angles on a Straight Line

Fact: Angles on a straight line add up to 180⁰

Example: Calculate angle a.

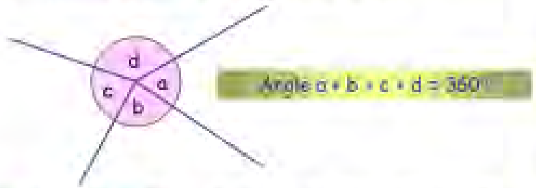


a = 180 - 50
= 130°

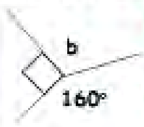
How to spot it: Find any continuous straight line, with another straight line joining it or cutting across it

Angles Around a Point

Fact: Angles around a point add up to 360⁰



Example: Calculate angle b.



Other ^ = 90 + 160
= 250°

b = 360 - 250
= 110°

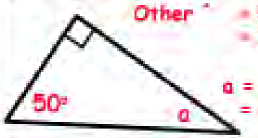
How to spot it: If you have a collection of lines all crossing at one point, then it's time to use this rule!

Angles in a Triangle

Fact: The interior (inside) angles of a triangle add up to 180⁰

a + b + c = 180⁰

Example: Calculate angle a



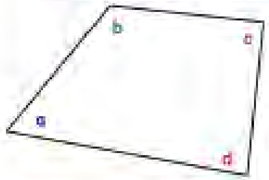
Other ^ = 50 + 90
= 140°

a = 180 - 140
= 40°

How to spot it: Find any type of triangle (equilateral, isosceles, right-angled, or scalene) and all the angles inside will add up to 180⁰

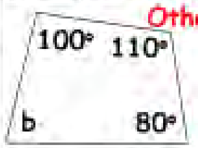
Angles in a Quadrilateral

Fact: Interior (inside) angles of a quadrilateral add up to 360⁰



a + b + c + d = 360⁰

Example: Calculate angle .



Other ^ = 100+110+80
= 290°

b = 360 - 290
= 70°

How to spot it: Find any 4 sided shape (square, rectangle, trapezium, kite, etc.) and the inside angles will add up to 360⁰

Mathematics

Higher

Unit 13

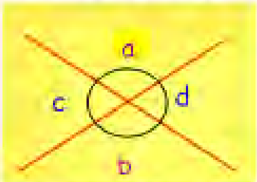
Parallel Lines

Parallel lines are lines which never meet, and always keep a perfectly equal distance apart.

Remember: Only assume lines are parallel if they have those little arrows on them:

Opposite Angles

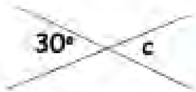
Fact: Opposite Angles are equal



a = b

c = d

Example: Calculate angle c.



c = 30°

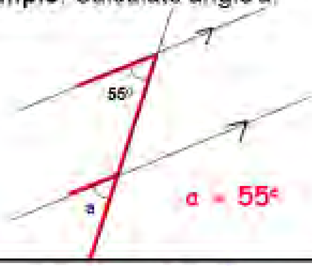
How to spot it: Find two continuous straight lines crossing at a point. The pairs of angles opposite each other will be equal

Note: Using Fact 2, all the angles around that point will add up to 360°

Corresponding Angles

Fact: Corresponding Angles are equal

Example: Calculate angle a.



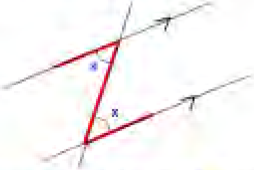
a = 55°

How to spot it: Look for the F shape, the angles underneath the arms of the F are equal

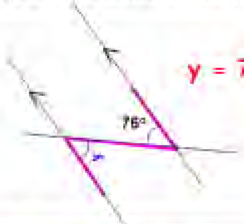
Note: The arms of the F must definitely be Parallel lines!

Alternate Angles

Fact: Alternate Angles are equal



Example: Calculate angle y.



y = 76°

How to spot it: Look for the Z shape, the angles "inside" the Z are equal

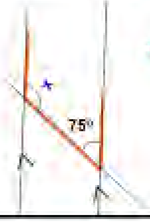
Note: The top and bottom of the Z must be Parallel Lines!

Interior Angles

Fact: Interior Angles add up to 180°

$x + y = 180°$

Example: Calculate angle x.



$x = 180 - 75$
$= 105°$

How to spot it: Look for the C shape, the angles underneath the top and bottom of the C add up to 180°

Note: The top and bottom of the C must definitely be Parallel lines!

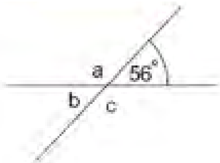
Mathematics

Higher

Unit 13

Example Questions

Example 1

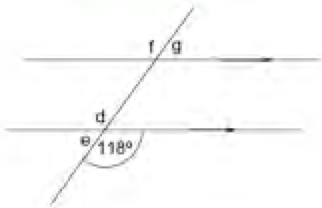


a = 180 - 56 = 124°
(Fact 1 - angles on a straight line)

b = 56°
(Fact 5 - opposite angles)

c = 360 - 56 - 124 - 56 = 124°
(Fact 2 - angles around a point)

Example 2



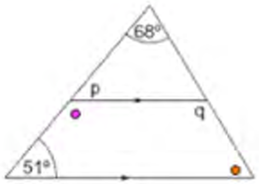
d = 118°
(Fact 5 - opposite angles)

e = 180 - 118 = 62°
(Fact 1 - angles on a straight line)

f = 118°
(Fact 6 - corresponding angles)

g = 180 - 118 = 62°
(Fact 1 - angles on a straight line)

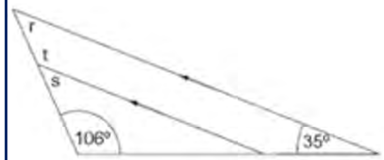
Example 3



p = 51°
(Fact 6 - corresponding angles)

To work out q:
● = 180 – 51 = 129° (Fact 1 – angles on a straight line)
● = 180 - 51 - 68 = 61° (Fact 3 – angles in a triangle)
q = 360 – 51 – 129 – 61 = 119°
(Fact 4 -angles in a quadrilateral)

Example 4

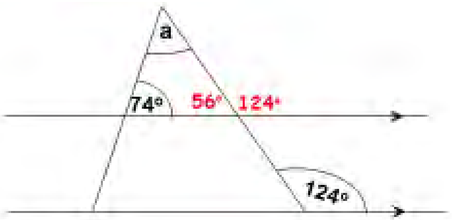


r = 180 – 106 – 35 = 39°
(Fact 3 – angles in a triangle)

s = 39°
(Fact 6 – corresponding angles)

t = 180 – 39 = 141°
(Fact 8 – interior angles)

Example 5

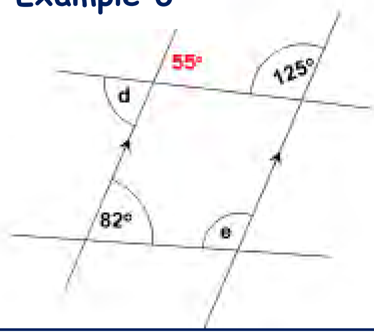


124 (Fact 6 - corresponding)

180 – 124 = 56° (Fact 1 – angles on a straight line)

a = 180 – (74 + 56) = 50° (Fact 3 – angles in a triangle)

Example 6



180 – 125 = 55° (Fact 8 - interior)

d = 55° (Fact 5 – opposite angles)

e = 180 – 82
= 98° (Fact 8 – interior angles)

Mathematics

Higher

Unit 13

Interior and Exterior Angles in Polygons

Key Words:

Polygon: The general term for a shape with any amount of sides.

Regular: A shape where all angles and sides are equal.

Irregular: A shape where the sides and angles are not all equal.

Interior Angles: The angles inside a shape.

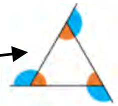
Exterior Angles: The angles outside a shape.

Angles in Polygons Rules (n is the number of sides in the polygon)

Sum of interior angles = $(n-2)\times 180°$

Sum of exterior angles = 360°

Interior angle + exterior angle = 180°



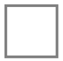
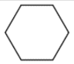
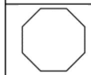
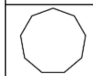
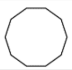
Additional Rules for Angles in a Regular Polygon

One interior angle = Sum of interior angles $\div n$

One exterior angle = $360 \div n$

$n = 360 \div$ one exterior angle

Regular Polygons

Shape	Name	Number of sides	Sum of interior angles	One interior angle	Sum of exterior angles	One exterior angle
	Equilateral Triangle	3	$(3-2)\times 180$ $= 180°$	$180 \div 3$ $= 60°$	360°	$360 \div 3$ $= 120°$
	Square	4	$(4-2)\times 180$ $= 360°$	$360 \div 4$ $= 90°$	360°	$360 \div 4$ $= 90°$
	Regular Pentagon	5	$(5-2)\times 180$ $= 540°$	$540 \div 5$ $= 108°$	360°	$360 \div 5$ $= 172°$
	Regular Hexagon	6	$(6-2)\times 180$ $= 720°$	$720 \div 6$ $= 120°$	360°	$360 \div 6$ $= 60°$
	Regular Heptagon	7	$(7-2)\times 180$ $= 900°$	$900 \div 7$ $= 128.6°$	360°	$360 \div 7$ $= 51.4°$
	Regular Octagon	8	$(8-2)\times 180$ $= 1080°$	$1080 \div 8$ $= 135°$	360°	$360 \div 8$ $= 45°$
	Regular Nonagon	9	$(9-2)\times 180$ $= 1260°$	$1260 \div 9$ $= 140°$	360°	$360 \div 9$ $= 40°$
	Regular Decagon	10	$(10-2)\times 180$ $= 1440°$	$1440 \div 10$ $= 144°$	360°	$360 \div 10$ $= 36°$

Finding the Number of Sides of a Regular Polygon

Example 1:

A regular polygon has exterior angles of 30°, how many sides does the polygon have?

Using the rule: $n = 360 \div$ one exterior angle

$n = 360 \div 30$

$n = 12$ The polygon has 12 sides.

Example 2:

A regular polygon has interior angles of 156°, how many sides does the polygon have?

Step 1: Using the rule: Interior angle + exterior angle = 180°

Rearrange to give: Exterior angle = 180 – Interior angle

= 180 – 156

= 24°

Step 2: Using the rule: $n = 360 \div$ one exterior angle

$n = 360 \div 24$

$n = 15$ The polygon has 15 sides.

Mathematics

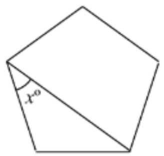
Higher

Unit 13

Regular Polygon Questions

Example 1:

The diagram shows a regular pentagon.
Work out the value of x.



Using the rule: Sum of interior angles = $(n - 2) \times 180°$

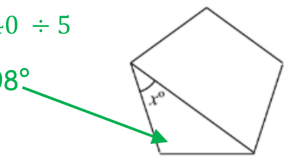
= $(5 - 2) \times 180°$

= 540°

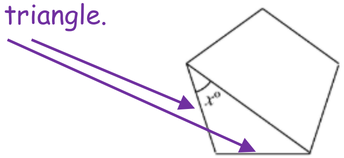
Using the rule: One interior angle = Sum of interior angles $\div n$

= $540 \div 5$

= 108°



As the shape is a regular shape, the 2 sides of the triangle must be equal, it is an isosceles triangle.

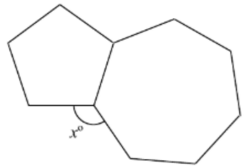


So, $x = \frac{180 - 108}{2}$

$x = 36°$

Example 2:

The diagram shows a regular pentagon and a regular heptagon.
Work out the value of x.



Angles around a point add to 360°, so x + one interior angle of the pentagon + one interior angle of the heptagon = 360°

Pentagon:

Using the rule: Sum of interior angles = $(n - 2) \times 180°$

= $(5 - 2) \times 180°$

= 540°

Using the rule: One interior angle = Sum of interior angles $\div n$

= $540 \div 5$

= 108°

Heptagon:

Using the rule: Sum of interior angles = $(n - 2) \times 180°$

= $(7 - 2) \times 180°$

= 900°

Using the rule: One interior angle = Sum of interior angles $\div n$

= $900 \div 7$

= 128.6° (1 d.p.)

$x = 360 - 108 - 128.6$

$x = 123.4°$

Mathematics

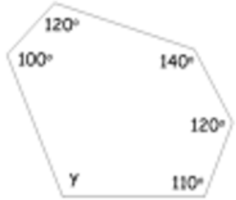
Higher

Unit 13

Irregular Polygon Questions

Example 1:

Find the size of angle y.



The shape has 6 sides, so it is a hexagon.

Using the rule: Sum of interior angles = $(n-2)\times 180°$

= $(6-2)\times 180°$

= 720°

Add up the interior angles we already have:

100 + 120 + 140 + 120 + 110 = 600

720 – 600 = 120° $y = 120°$

Example 2:

Four of the interior angles of a seven-sided polygon are 114°, 150°, 160° and 170°. The other three interior angles of this polygon are equal. Calculate the size of each of the other three interior angles.

Using the rule: Sum of interior angles = $(n-2)\times 180°$

= $(7-2)\times 180°$

= 900°

The four interior angles add to: 114 + 150 + 160 + 170 = 584°

900 – 584 = 306°

306 ÷ 3 = 102° Each of the other three interior angles is 102°.

Example 3:

Two of the exterior angles of a hexagon are 110° and 130°. The other exterior angles are all equal. Calculate the size of the largest of the interior angles of this hexagon.

Note: Take care not to get confused, this question talks about exterior angles AND interior angles.

Using the rule: Sum of exterior angles = 360°

360 – (110 + 130) = 120°

120 ÷ 4 = 30°

The other interior angles are all 30°

Note: The smallest exterior angles will give the largest interior angles.

Using the rule: Interior angle + exterior angle = 180°

Rearrange to give: Interior angle = 180 – Exterior angle

= 180 – 30

= 150°

The largest of the interior angles is 150°.

Mathematics

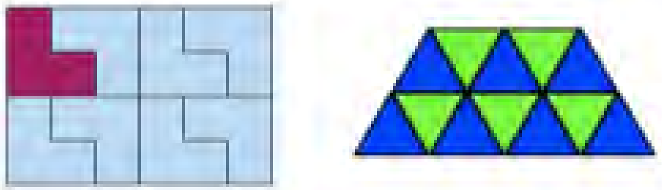
Higher

Unit 13

Tessellation

A tessellation is a pattern created with identical shapes that fit together with no gaps.

These shapes tessellate – they fit together with no gaps between them.

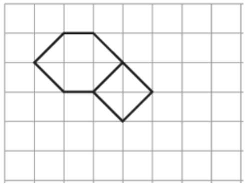


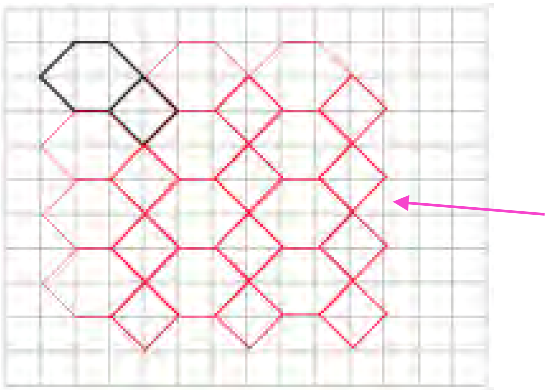
These shapes do not tessellate – when they are put together, they have gaps between them.

Regular polygons tessellate if the interior angles can be added together to make 360° (a full turn), i.e. if one interior angle is a factor of 360.

Example 1:

Ben needs to tile his kitchen floor and decides to use the two types of tiles shown in the diagram. By drawing more tiles in the diagram, show that the tiles will tessellate.

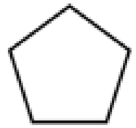




The shapes fit together with no gaps.

Example 2:

Shown is a regular pentagon. Will the regular pentagon tessellate? You must show your workings.



Using the rule: Sum of interior angles = $(n-2)\times 180°$

= $(5-2)\times 180°$

= 540°

Using the rule: One interior angle = Sum of interior angles $\div n$

= $540 \div 5$

= 108°

Is 108° a factor of 360? $\frac{360}{108} = 3.3$

108° is not a factor of 360°, therefore a regular pentagon will not tessellate.

Mathematics

Higher

Unit 14

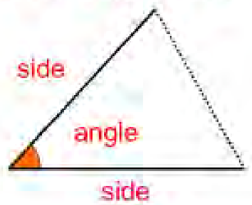
Construction

Construction is the act of drawing shapes, angles or lines accurately using a compass, protractor, and a ruler.

Constructing SAS Triangles

How could we construct a triangle given the lengths of two of its sides and the angle between them?

Side Angle Side



The angle between the two sides is often called the **included angle**.

Example: Construct triangle ABC with $AB = 6\text{cm}$, $\hat{B} = 68°$, and $BC = 5\text{cm}$.

Step 1: Start by drawing side AB with a ruler.

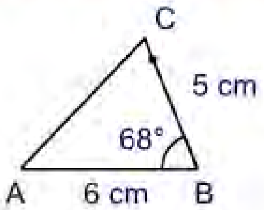
Step 2: Use a protractor to mark an angle of 68° from point B.

Step 3: Use a ruler to draw a line of 5 cm from B to C.

Step 4: Join A to C using a ruler to complete the triangle.

Check your accuracy by measuring the length of AC.

6.3 cm



Constructing ASA Triangles

How could we construct a triangle given two angles and the length of the side between them?

Angle Side Angle

The side between the two angles is often called the **included side.**

Example: Construct triangle ABC with $AB = 5\text{cm}$, $\hat{A} = 35°$, and $\hat{B} = 115°$.

Step 1: Start by drawing side AB with a ruler.

Step 2: Use a protractor to mark an angle of 35° from point A.

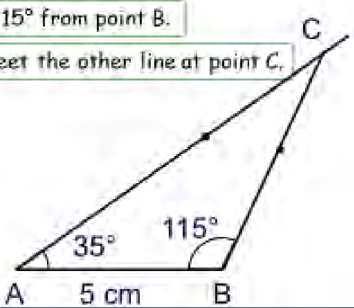
Step 3: Use a ruler to draw a long line from A.

Step 4: Use a protractor to mark an angle of 115° from point B.

Step 5: Use a ruler to draw a line from B to meet the other line at point C.

Check your accuracy by measuring the length of AC and BC.

9.0 cm 5.7 cm



Mathematics

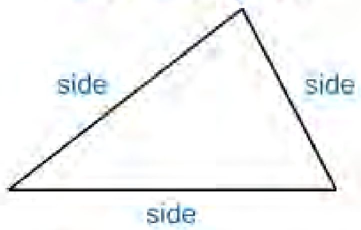
Higher

Unit 14

Constructing SSS Triangles

How could we construct a triangle given the lengths of three sides using a compass and ruler?

Side Side Side



To construct this angle you will need to use a compass.

Example: Construct triangle ABC with $AB = 4$cm, $AC = 5$cm, and $BC = 3$cm.

Step 1: Start by drawing side AB with a ruler.

Step 2: Open a pair of compasses to a length of 5cm.

Step 3: Put the compass needle at point A and draw an arc above line AB.

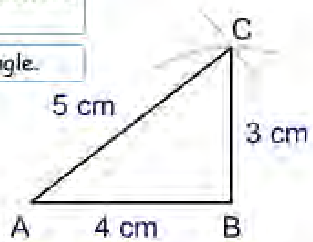
Step 4: Next, open the compasses out to a length of 3cm.

Step 5: Put the compass needle at point B and draw an arc crossing over the other one. This is point C.

Step 6: Draw lines AC and BC to complete the triangle.

Check your accuracy by measuring angle C.

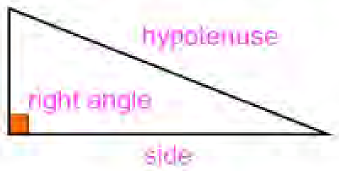
54°



Constructing RHS Triangles

How could we construct a right-angled triangle given the right angle, the length of the hypotenuse and the length of one other side?

Right angle Hypotenuse Side



Remember, the longest side in a right-angled triangle is called the **hypotenuse**.

Example: Construct triangle ABC with $AB = 5$cm, $\hat{B} = 90°$, and $AC = 7$cm.

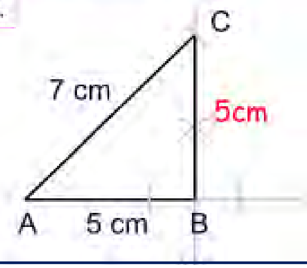
Step 1: Start by drawing side AB with a ruler.

Step 2: Extend AB and use compasses to construct a perpendicular at point B.

Step 3: Open the compasses to 7cm.

Step 4: Place the compass needle on A and draw an arc on the perpendicular.

Step 5: Label this point C and complete the triangle.



Mathematics

Higher

Unit 14

Constructing quadrilaterals

Example: Construct the quadrilateral $ABCD$ with $AB = 6\text{cm}$, $\hat{A} = 100°$, $BD = 2\text{cm}$, and $\hat{B} = 120°$.

Step 1: Start by drawing side AB with a length of 6cm using a ruler.

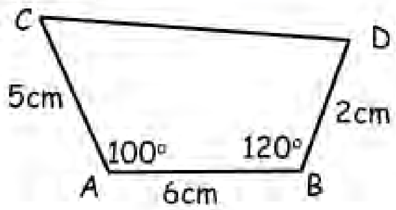
Step 2: Use a protractor to mark an angle of 100° from point A.

Step 3: Use a ruler to draw a line 5cm from A going through the marked point in step 2 and label it C.

Step 4: Use a protractor to mark an angle of 120° from point B.

Step 5: Use a ruler to draw a line 2cm from B going through the marked point in step 4 and label it D.

Step 6: Draw a line connecting C and D.



Check your accuracy by measuring length CD.

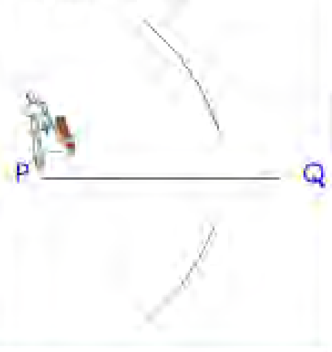
8.5cm

Constructing a Perpendicular Bisector (90°)

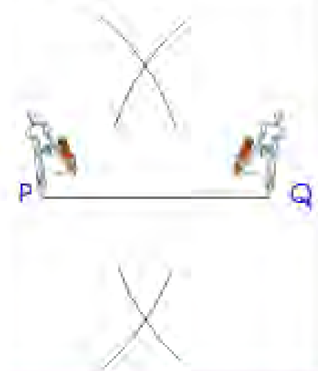
A perpendicular bisector is a line that cuts a line segment at 90° into two equal parts (in half).

Example: Construct a perpendicular bisector to the line PQ

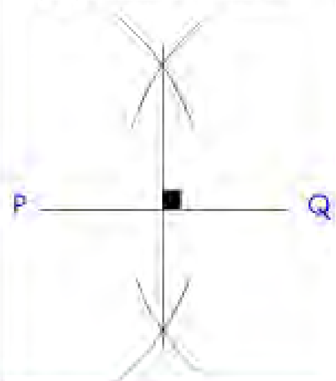
1. Set your compass to over half the length of the line. Place the pointy bit of the compass at **P** and draw an arc above and below the line:



2. Making sure you keep your compass at the exact same setting, place the pointy bit at **Q** and draw two more arcs.



3. With your ruler, draw a straight line through the two points where the arcs cross, and that is your perpendicular bisector!



Note: Every point on this new line is the same distance from point P as point Q

Mathematics

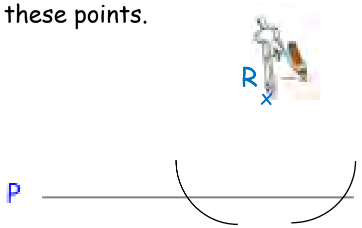
Higher

Unit 14

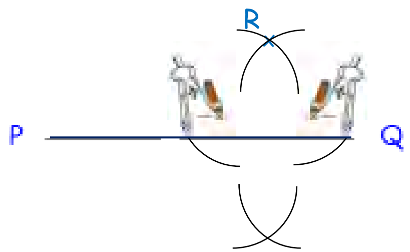
Constructing a Perpendicular Bisector from a Point to a Line

Example: Construct a perpendicular line from the point R to the line PQ

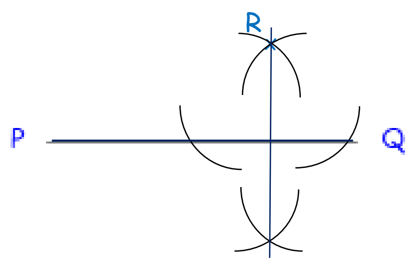
1. Put the point of your compass on point R. Open the compass so that it will cross the line PQ in two places, draw an arc at each of these points.



2. Put the point of your compass where each of the arcs crosses the line PQ. Draw an arc above and below the line. The arc should go through point R.



3. With your ruler, draw a straight line through the two points where the arcs cross, that is your perpendicular bisector from point R to the line PQ.



Constructing an Angle Bisector

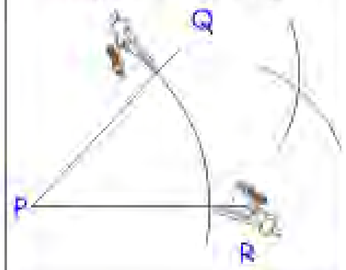
An angle bisector is a line that cuts an angle into two equal angles (in half).

Example: Construct an angle bisector for the angle made by lines PQ and PR

1. Place the pointy bit of your compass at P and draw an arc which crosses lines PQ and PR

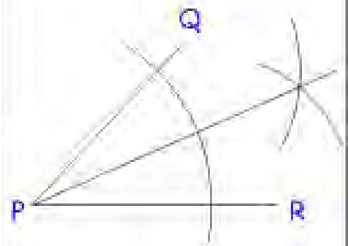
2. Place the pointy bit of the compass at both of the places where the arc hits the lines and draw two arcs

Crucial: You must not change the setting of the compass at this stage!



3. With your ruler, draw a straight line from P through the intersection of the arcs.

This is your angle bisector!



Note: Every point on this line is the same distance from line PQ as line PR

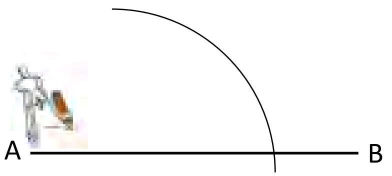
Mathematics

Higher

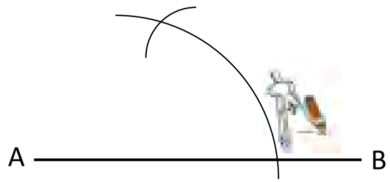
Unit 14

Constructing a 60° Angle

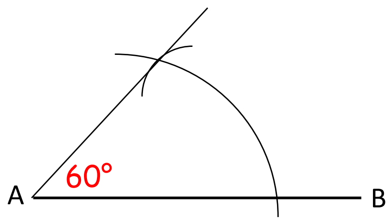
1. If you do not have a base line then draw one and label the ends A and B. Place the point of the compass at A, open the compass up more than half-way along the line, draw an arc that goes through the line.



2. Without moving the setting of the compass, place the point of the compass where the arc crosses the line. Draw another arc that crosses the first arc.



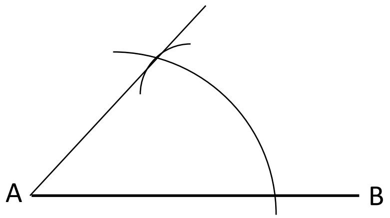
3. With your ruler, draw a straight line that joins the point where the two arcs cross to A. This angle is 60°.



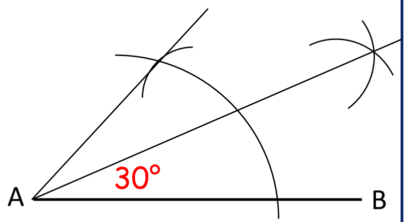
Constructing a 30° Angle

We can use the fact that 30° is half of 60°.

1. Construct a 60° angle.



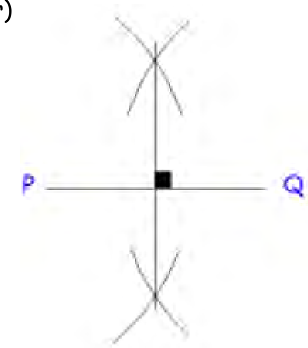
2. Bisect the angle. Label the new angle 30°.



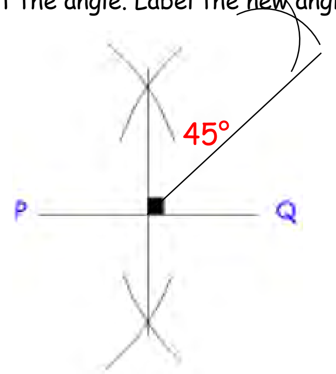
Constructing a 45° Angle

We can use the fact that 45° is half of 90°.

1. Construct a 90° angle (perpendicular bisector)



2. Bisect the angle. Label the new angle 45°.



Mathematics

Higher

Unit 14

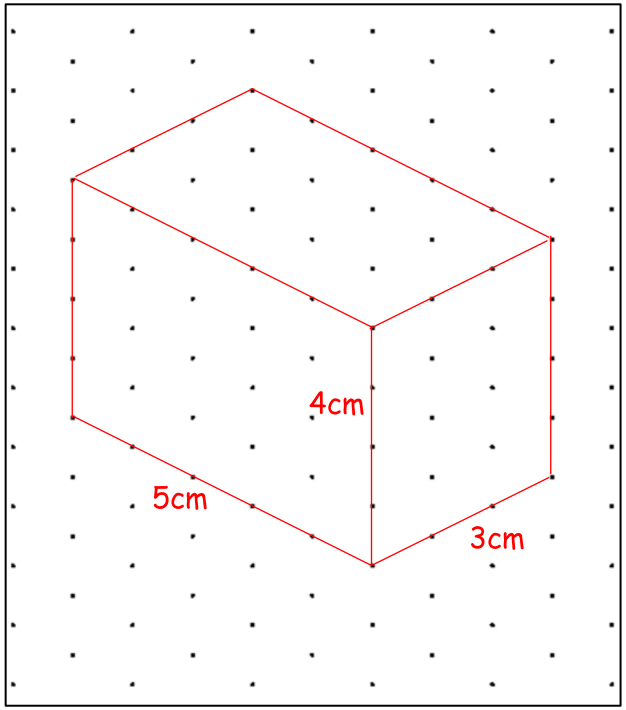
Isometric Drawing

We can draw 3-D shapes on isometric paper (dotted paper).

Example: Draw an isometric representation of a cuboid measuring 5cm by 4cm by 3cm.

The isometric paper has 1cm spaces between each dot on the diagonal. To draw the cuboid to scale we need to use lines along the diagonal dots.

Remember, when you are drawing the line you are counting the spaces between each dot, not the number of dots. So, for a line of 5cm you will count 5 spaces long.



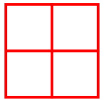
Plans and Elevations

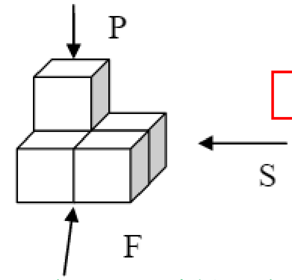
Plans and elevations are 2-D drawings of 3-D shapes.

A plan is the view of a 3-D shape when you are looking down onto an object from above.

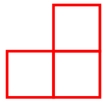
An elevation is the view of a 3-D shape when you are looking at it from the front or from the side.

A plan of this shape would be the view looking down onto it

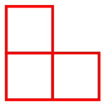




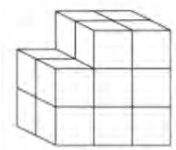
A side elevation of this shape would be the view looking at the shape from the side



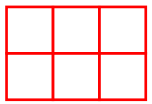
A front elevation of this shape would be the view looking at the shape from the front



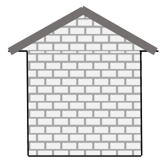
Example 1: Draw a sketch of the plan view of the shape below.



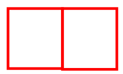
This is what the shape would look like from above



Example 2: Draw a sketch of the plan view of the house.



This is what the house would look like from above



Mathematics

Higher

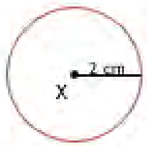
Unit 15

Loci

Loci/Locus is the path that an object moves along under certain conditions.

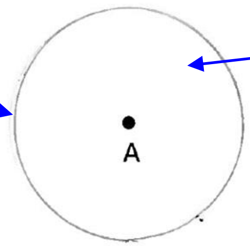
There are four main loci

1) **Locus of one point**: We can recognise it by the use of a single letter, e.g. A or B. We draw a circle.

The locus of points 2 cm from a point X on a flat surface (2D)	
Is a circle with radius 2 cm, centre X.	 The locus less than 2 cm from X will be the inside of the circle.
The locus of points 4 cm from a point X in space (3D)	
	Is a sphere with radius 4 cm, centre X.

Locus of points equidistant from a point

Draw all the points which are 2.5cm from the point A.



The locus of points equidistant from a point is a __Circle____.

Every point on the circle is 2.5cm away from A.

The locus of points < 2.5cm (less than 2.5cm) would be everything inside the circle.

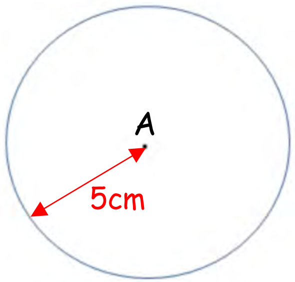
The locus of points > 2.5cm (greater than 2.5cm) would be everything outside the circle.

Method

Example 1: Draw all the points which are 5cm away from the point A.

– Open your compass to the length specified in the question 5cm.

– Place the point of your compass on A, draw a circle.

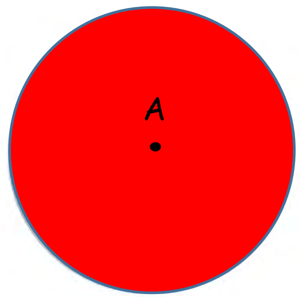


Example 2: Shade the region that is < 5cm away from the point A.

– Open your compass to the length specified in the question 5cm.

– Place the point of your compass on A, draw a circle.

– Everything inside the circle is < 5cm from A; shade inside the circle.

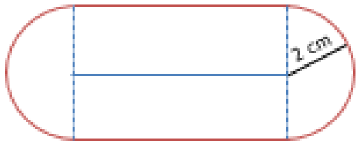


Mathematics

Higher

Unit 15

2) **Locus of one line**: We can recognise it by the use of two letters together, e.g. AB or PQ. We draw parallel line/lines (or a racing track).

The locus of points 2 cm away from a line	
Has a line 2cm directly above and below the line and a semicircle with radius 2 cm on each end of the line.	

Method

Example: Draw the locus of all the points 1cm from line AB.

– Open your compass to the length specified in the question 1cm.

– Draw a semi-circle around point A

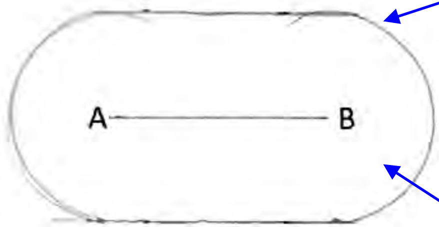
– Keeping the compass the same size, repeat for point B

– Measure 1cm above the line AB and 1cm below the line AB. Join these points to the semi-circles using a ruler.

All these steps are shown in the diagram in red.

Locus of points equidistant from a line

Draw all the points which are 2cm from the line AB.



Every point on the line is 2cm away from the line AB.

The locus of points < 2cm (less than 2cm) from the line AB would be everything inside the racing track.

The locus of points > 2cm (greater than 2cm) from the line AB would be everything outside the racing track.

The locus of points equidistant from a line is two Parallel lines and two Semi-circles.

It looks like a Racing Track.

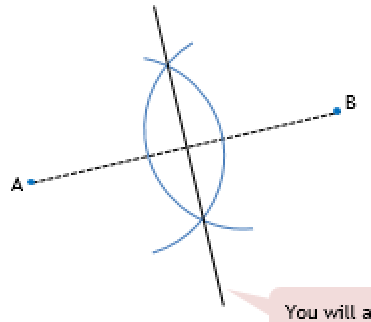
Mathematics

Higher

Unit 15

3) **Locus of two points**: We can recognise it by the use of two lots of single letters, e.g. A and B or P and Q. We draw a perpendicular bisector.

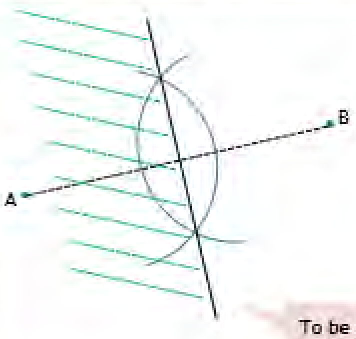
The locus of points equidistant from 2 points



The locus equidistant from 2 points is the perpendicular bisector of the line that joins the 2 points. (Draw the joining line if it helps.)

You will always be the same distance from both A and B from anywhere on this bisector.

The locus of points closer to A than B



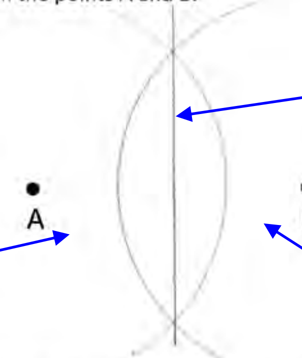
- Bisect the line between the two points. This is the boundary, on one side of the boundary you are closer to A and on the other side you are closer to B.
- Shade the side of the boundary that satisfies the required condition.

To be closer to one point than the other means you will need to be on one side of the boundary.

The perpendicular bisector cuts the 2 points in half at 90°

Locus of points equidistant from two points

Draw all the points which are equidistant from the points A and B.



The locus of points equidistant from two points is the Perpendicular Bisector.

Every point on the line is equidistant from A and B

The locus of points closer to A than B would be anything on the left hand side of the line

The locus of points closer to B than A would be anything on the right hand side of the line

Method

Example: Bisect line AB show.

(or it could be asked as, *draw a line equidistant from points* A *and* B, the method will be exactly the same_

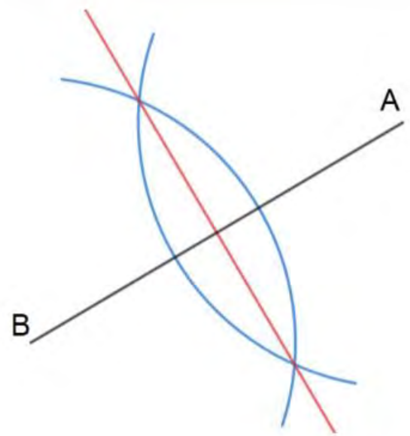
To do this you must do the following:

– Set your compasses to a fixed length apart (**Must be greater then half the line**)

– Put your compass on point A and draw an arc (blue).

– With your compasses at the **same length**, repeat step 2 for the other end of the line (also blue).

– Then, draw a line which passes through the two crossing points. This line (red) is the perpendicular bisector.



Mathematics

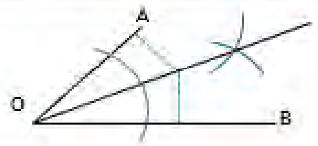
Higher

Unit 15

4) **Locus of two lines**: We can recognise it by the use of two lots of two letters together, e.g. AB and AC or PQ and QR (there will be a common letter). We draw an angle bisector.

The angle bisector cuts the angle between the 2 lines in half.

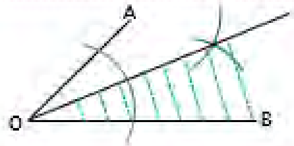
The locus of points equidistant from 2 lines



You will always be the same distance from each line by the shortest route. (The shortest route is perpendicular to each line.)

The locus equidistant from the lines OA and OB is the bisector of the angle between the 2 lines.

The locus of points closer to line OB than OA

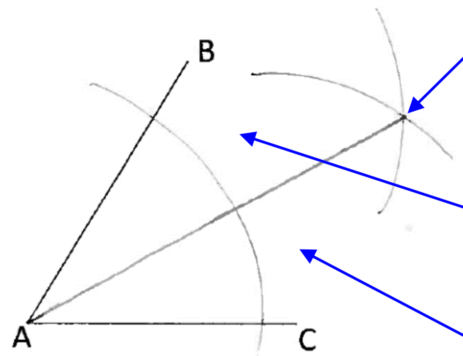


To be closer to one line than the other means you will need to be on one side of the boundary.

- Bisect the angle between the lines. This is the boundary.
- Shade the side of the boundary that satisfies the required condition.

<u>Locus of points equidistant from two lines</u>

Draw all the points which are equidistant from the lines AB and AC.



The locus of points equidistant from two lines is the <u>Angle Bisector</u>.

Every point on the line is equidistant from lines AB and AC.

The locus of points closer to AB than AC would be anything on the left hand side of the line

The locus of points closer to AC than AB would be anything on the right hand side of the line.

Method

Example: Construct a line, equidistant from line AB and BC

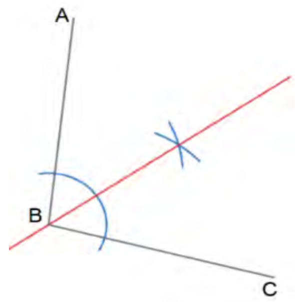
In the diagram, we have been given two lines (black) and an angle between them. To bisect it, we do the following:

– Place your compass on the corner where the two lines meet and draw an arc (**blue**) that passes through both lines.

– Place your compass on the crossing point and draw a small arc (**blue**) between the lines.

– With your compass at the **same length**, repeat step 2 from the other crossing point.

– Draw a line (red) passing through the corner where the lines meet and the point where the two green arcs cross. This is the angle bisector.



Mathematics

Higher

Unit 15

Note: When answering Loci questions, highlight any key words and work out what construction type is needed for each step.

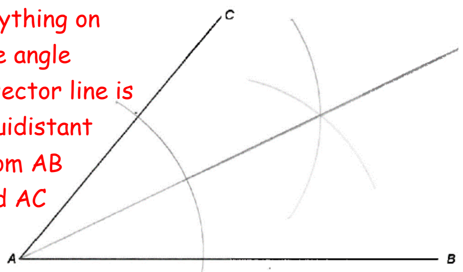
Example 1: Two straight lines AB and AC are shown below. The point P is:

- Equidistant from line AB and AC — angle bisector
- 5cm from C — circle 5cm from C
- More than 7cm from A — circle 7cm from A

Show clearly the position of point P

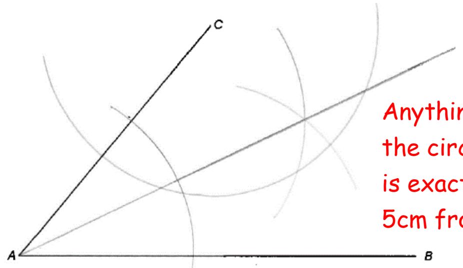
Step 1: Draw an angle bisector

Anything on the angle bisector line is equidistant from AB and AC



Point P must be somewhere on the angle bisector line

Step 2: Draw a 5cm circle from C



Anything on the circle line is exactly 5cm from C

Point P must be somewhere on the angle bisector line…

…and on the circle line.

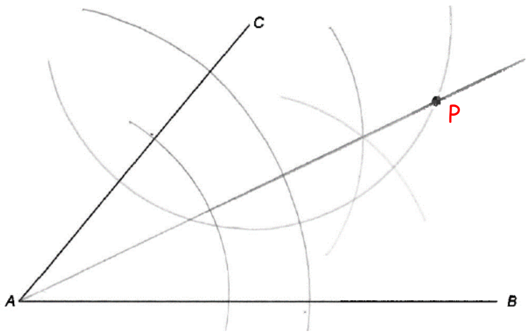
It must be one of the two places where the two lines meet.

Step 3: Draw a 7cm circle from A

Point P must be somewhere on the angle bisector line…

…and on the circle line…

…and outside the 7cm circle

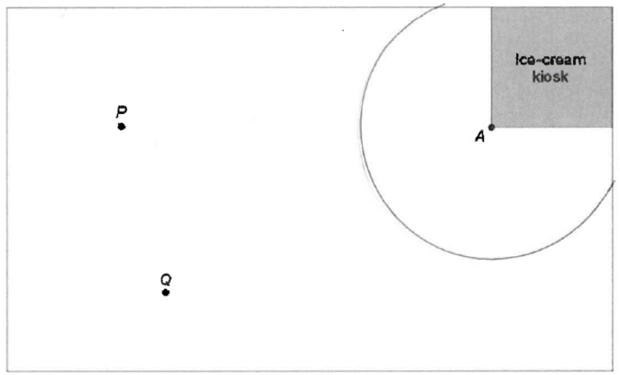


Example 2: When Fisher tours takes a coach trip to the zoo, they must follow instructions for parking:

- Must be further than 12 metres away from point A at the ice-cream kiosk; — Circle 3cm from A, using scale 1cm = 4m
- Must be closer to the lamp post Q than to the lamp post P. — Perpendicular bisector

Indicate, on the scale drawing above, the region in which Fisher Tours can park their coach.

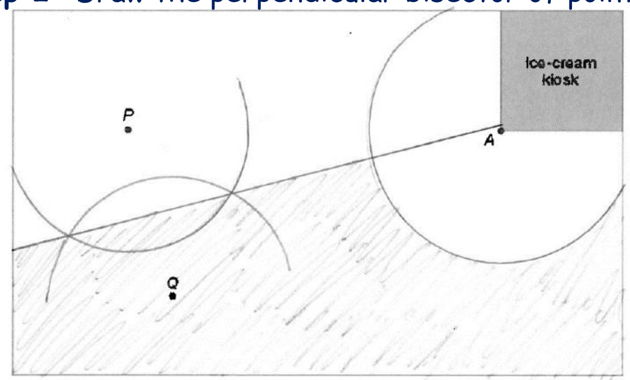
Step 1: Draw a 3cm (12m) circle at point A



Anything outside the circle is more than 12 meters from point A

Scale: 1cm represents 4 metres.

Step 2: Draw the perpendicular bisector of point P and Q



Anything to the right of the perpendicular bisector is nearer to Q than to P

Parking must be outside the circle…

…and to the right of the perpendicular bisector

Scale: 1cm represents 4 metres.

Mathematics

Higher

Unit 15

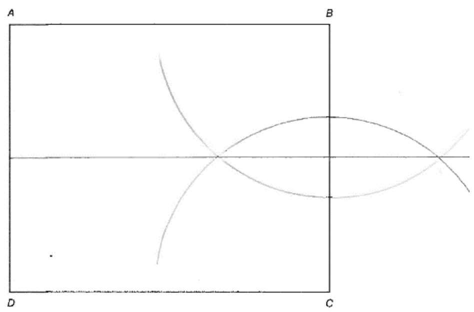
Example 3: The diagram shows rectangle $ABCD$.
Indicate the region that is inside the rectangle $ABCD$ and is also:

- Closer to point C than to point B — perpendicular bisector
- Closer to line AD than to line AB — angle bisector
- More than 6cm from D — circle 6cm from D

Show clearly the position of point P

Step 1: Construct the perpendicular bisector of BC

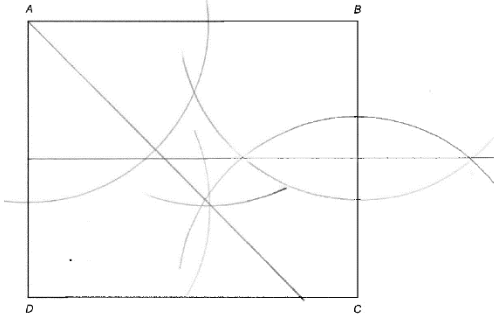
Anything below the perpendicular bisector line is nearer to C than to B



The region to shade is inside the rectangle and closer to point C than to point B

Step 2: Construct an angle bisector at A

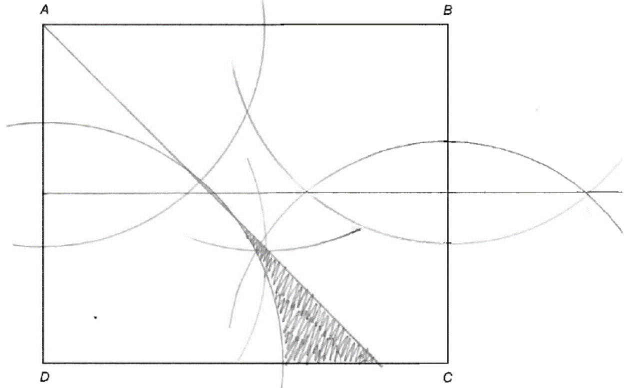
Anything below the angle bisector line is nearer to AD than to AB



The region to shade is inside the rectangle, closer to point C than to point B…
…and closer to AD than to AB

Step 3: Construct a 6cm circle from point D

Anything outside the 6cm circle is more than 6cm from D



The region to shade is inside the rectangle, closer to point C than to point B, closer to AD than to AB …
…and outside the 6cm circle

Example 4: Shade the region inside the triangle that satisfies both of the following conditions:

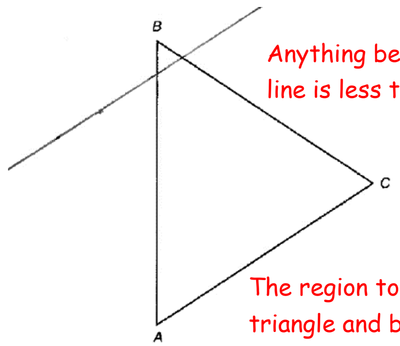
- It is less than 5cm from AC, and

  line parallel to and 5cm from AC

- It is less than 4cm from B

  4cm circle from B

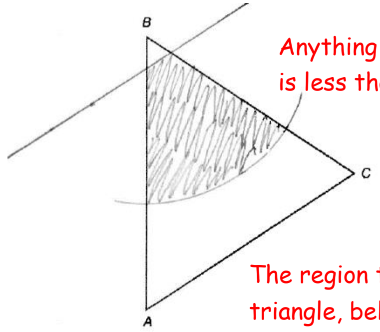
Step 1: Construct a line parallel to and 5cm from AC



Anything below the parallel line is less than 5cm from AC

The region to shade is inside the triangle and below the parallel line

Step 2: Construct a 4cm circle from B



Anything inside the 4cm circle is less than 4cm from B

The region to shade is inside the triangle, below the parallel line…
…and inside the 4cm circle

Mathematics

Higher

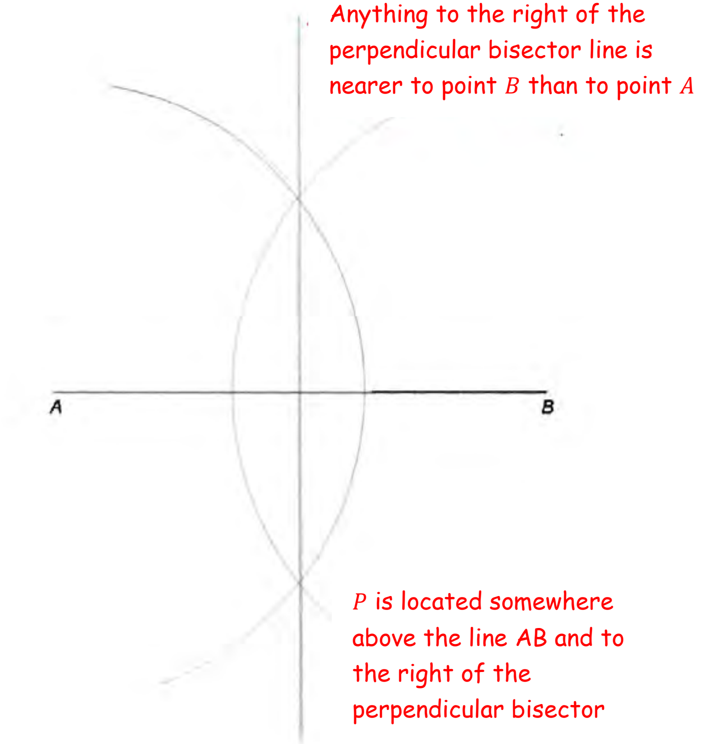
Unit 15

Example 5: The line AB is drawn below.
The point P lies above the line AB.
The region in which P is located is such that:

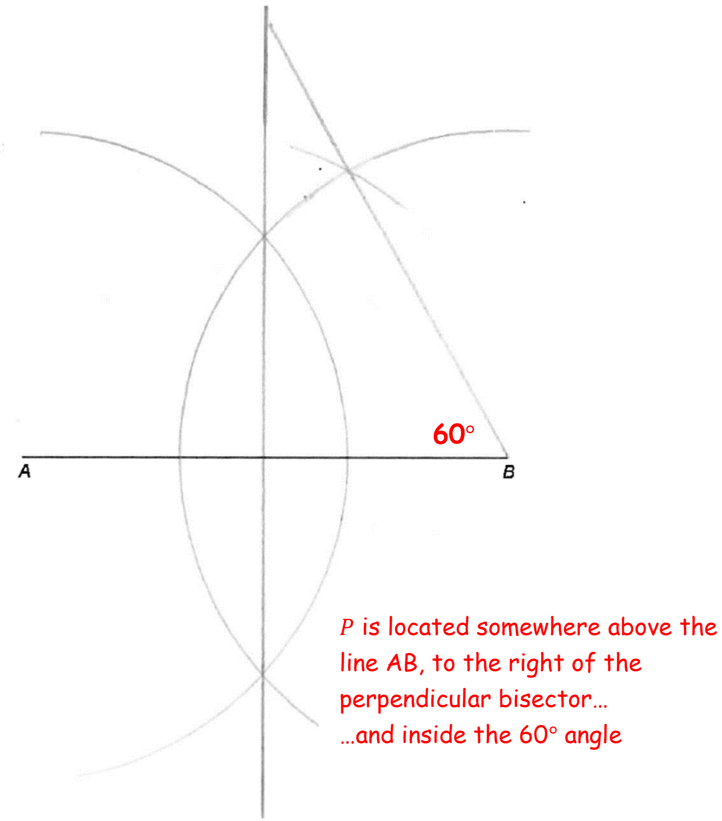
- P is nearer to point B than to point A perpendicular bisector
- $A\hat{B}P \leq 60°$ 60° angle
- $BP \geq 6cm$ circle 6cm from B

Using a ruler and a pair of compasses, construct suitable lines and arcs to represent these conditions.
Construction arcs must be clearly shown.
Shade the region in which the point P is located.

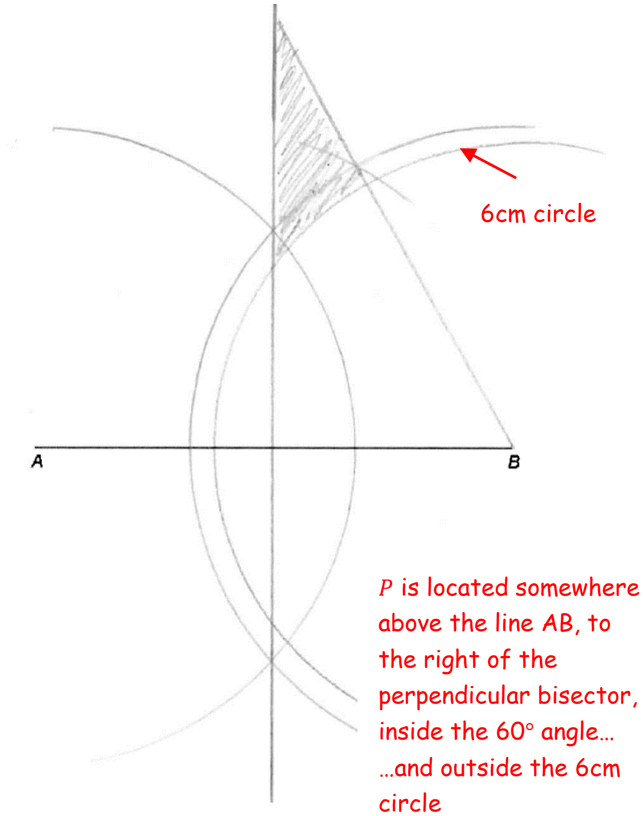
Step 1: Construct the perpendicular bisector



Step 2: Construct a 60° angle



Step 3: Construct a 6cm circle from point B, shade the region where point P is located



Mathematics

Higher

Unit 16

Substitution in Algebra

Substitution is where you are told the value of a letter and you substitute this into an expression or equation.

e.g. Find the value of $5x$ when $x = 7$, means 5 x x = 5 x 7 = 35.

- Always apply BIDMAS/BODMAS
- Use brackets for powers
- For fractions, work out the top and bottom separately.

Example 1: Evaluate (find the **value** of) the expressions, given that:

$$a = 2,\ b = 3,\ c = -5, d = -1$$

a) $5a = 5 \times 2$
$= \mathbf{10}$

b) $3b - 2c = 3 \times 3 - 2 \times (-5)$
$= 9 + 10$
$= \mathbf{19}$

c) $4b^2 + d = 4 \times 3^2 + (-1)$
$= 4 \times 9 - 1$
$= 36 - 1$
$= \mathbf{35}$

d) $3a^3 = 3 \times (2)^3$
$= 3 \times 8$
$= \mathbf{24}$

e) $\frac{5cd}{a+b} = \frac{5 \times (-5) \times (-1)}{2+3}$
$= \frac{25}{5}$
$= \mathbf{5}$

f) $c^2 + abd = (-5)^2 + 2 \times 3 \times (-1)$
$= 25 - 6$
$= \mathbf{19}$

Example 3: Use the formula $P = 5A - 6B$ to find the value of:

a) P when $A = 7$ and $B = -4$.

$P = 5A - 6B$

$P = 5 \times 7 - 6 \times (-4)$

$P = 35 + 24$

$P = 59$

b) A when $B = 3$ and $P = 37$

$P = 5A - 6B$

$37 = 5A - 6 \times 3$

$37 = 5A - 18$

$37 + 18 = 5A$

$55 = 5A$

$\frac{55}{5} = A \qquad A = 11$

Example 2: Evaluate (find the **value** of) the expressions, given that: (*calculator questions*)

$$a = 1.2,\ b = \frac{1}{9},\ c = -3.65$$

a) $4b - 6c + a^2 = 4 \times \frac{1}{9} - 6 \times (-3.65) + (1.2)^2$
$= \frac{4}{9} + 21.9 + 1.44$
$= \mathbf{23.78\dot{4}}$

b) $\sqrt{\frac{a+4c}{3b+c}} = \sqrt{\frac{1.2 + 4 \times (-3.65)}{3 \times \frac{1}{9} + (-3.65)}}$

$= \sqrt{\frac{-13.4}{-3.31\dot{6}}}$

$= \sqrt{4.0402010051}$

$= 2.0100251255$

Learn how to do these in one step using your *scientific* calculator.

Mathematics

Higher

Unit 16

Example 4: A security firm uses the following formula to give the approximate number of staff it will need for certain events:

$$N = 0.035A + \frac{d^2}{300}$$

N is the number of staff needed.
A is the estimated number of people attending the event.
d is a measure related to the area that will need to be patrolled.

How many staff will be needed at an event where the estimated attendance is 550 and d is given as 50? Give your answer correct to the nearest whole number.

$$N = 0.035A + \frac{d^2}{300}$$

$$N = 0.035 \times 550 + \frac{50^2}{300}$$

$$N = 27.5833 \ldots$$

So, $N = 28$ staff (to nearest whole number).

Example 5: Helen makes greeting cards which she sells at a weekly market. Her weekly profit (P), in pounds, is given by the formula:

$$P = 2.99S - 0.7M$$

Where S is the number of cards she sells and M is the number of cards she made.

One week she sold 60 cards but made a loss of £30.60.
How many cards had she made?

$S = 60$ and $P = -30.60$

$$P = 2.99S - 0.7M$$

$$-30.60 = 2.99 \times 60 - 0.7M$$

$$-30.60 = 179.4 - 0.7M$$

$$-210 = -0.7M$$

$$-\frac{210}{-0.7} = M$$

So, $M = 300$

Example 6: A gas company uses the following formula to calculate how much to charge its customers:

$$charge\ (in\ pence) = (U \times 11.546 + D \times 31.48) \times 1.05$$

The number of units of gas used by a customer is U and the number of days in the billing period is D.

A customer was charged £165.53 over a billing period of 90 days.
Calculate the number of gas units this customer used during this period.

$$charge\ (in\ pence) = (U \times 11.546 + D \times 31.48) \times 1.05$$

£165.53 = 16553p

$$16553 = (U \times 11.546 + 90 \times 31.48) \times 1.05$$

$$\frac{16553}{1.05} = (U \times 11.546 + 90 \times 31.48)$$

$$\frac{16553}{1.05} = 11.546U + 2833.2$$

$$\frac{16553}{1.05} - 2833.2 = 11.546U$$

$$\frac{12931.5610\ldots}{11.546} = U$$

$$1120.0036 \ldots = U$$

So, $U = 1120$ gas units used.

Mathematics

Higher

Unit 16

Function Machines / Number Machines

Example 1:

A number machine is shown below.

INPUT → $\div 2$ → -8 → OUTPUT

a) Calculate the OUTPUT when the INPUT is 10.

Start with 10

Divide by 2 to give 5

Subtract 8 to give -3 — The OUTPUT is -3

b) Calculate the INPUT when the OUTPUT is 7.

To find the INPUT from the OUTPUT use the inverse operations

Start with 7

Add 8 to give 15

Multiply by 2 to give 30 — The INPUT is 30

c) Write down an expression for the OUTPUT when the INPUT is n.

Start with n

Divide by 2 to give $\frac{n}{2}$

Then subtract 8 to give $\frac{n}{2} - 8$ — The OUTPUT is $\frac{n}{2} - 8$

Example 2:

A number machine is shown below.

INPUT → $+2$ → $\times 4$ → OUTPUT

a) Calculate the OUTPUT when the INPUT is 3.

Start with 3

Add 2 to give 5

Multiply by 4 to give 20 — The OUTPUT is 20

b) Calculate the INPUT when the OUTPUT is 16.

To find the INPUT from the OUTPUT use the inverse operations

Start with 16

Divide by 4 to give 4

Subtract 2 to give 2 — The INPUT is 2

c) Write down an expression for the OUTPUT when the INPUT is n.

Start with n

Add 2 to give $n + 2$

Multiply everything by 4 to give $4(n + 2)$ — The OUTPUT is $4(n + 2)$

Mathematics

Higher

Unit 17

Solving Linear Equations

A linear equation is an equation (has an equals sign) involving letters and numbers, where the highest power of any letter is 1.
The aim of solving an equation is to find the value of the unknown which makes the equation balance, e.g. equation: $x - 5 = 3$, solution: $x = 8$, because $8 - 5 = 3$.

There are different methods you can use to solve equations using your knowledge of inverse operations.

An operation is a mathematical process such as adding, multiplying, or squaring, etc.

An inverse operation is the process of reversing the operation (the opposite process). For example, when adding, the inverse operation would be subtracting, when multiplying the inverse operation would be division and so on.

Here are the main inverse operations you need to know:

$+$ ⇄ $-$

Addition is the opposite of subtracting.
Subtracting is the opposite of adding.
They are inverse operations.

$\times$ ⇄ $\div$

Multiplication is the opposite of division. Division is the opposite of multiplication
They are inverse operations.

$\sqrt{}$ ⇄ 2 (power of 2)

Square rooting is the opposite of squaring. Squaring is the opposite of square rooting.
They are inverse operations.

Method 1: Using our knowledge of inverse operations we can rearrange the equation to get the letter (this is often x) on its own. Many teachers say this is called the "Change the side, change the sign" method.

Golden Rule: When rearranging an equation and moving a term over the equals sign to the **opposite side** it changes to the **opposite sign** (the inverse). For example, '+3' becomes '-3', or '÷4' become 'x4'.

Note: The subject term is the letter used in the equation.

Step 1: Get rid of any square root signs by squaring both sides. Clear any fractions by cross-multiplying up to every other term. Multiply out any brackets.

Step 2: Collect all subject terms on one side of the equals sign and all non-subject terms on the other. Remembering the rule "change sides, change sign" (you most often see the letters on the left-hand side and numbers on the right).

Step 3: Simplify like terms on each side of the equation.

Step 4: If you are left with a number multiplied by your subject term equals something ($Ax = B$ where A and B are numbers and x is the subject term), then to get the subject term on its own, move the number over the other side of the equals sign remembering to change its sign to the opposite sign (the inverse) which in this case is from a multiply to a divide ($Ax = B$ becomes $x = \frac{B}{A}$).

Check your answer using <u>substitution</u> to make sure you are right.

Mathematics

Higher

Unit 17

Example 1: $7p - 3 = 32$

$$7p - 3 = 32$$

Move the -3 over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$$7p = 32 + 3$$

$$7p = 35$$

Remember, $7p$ means $7 \times p$

Move the $\times 7$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$$p = \frac{35}{7}$$

Remember, $\frac{35}{7}$ means $35 \div 7$

$$p = 5$$

Example 2: $2(3r + 6) = 36$

$$2(3r + 6) = 36$$

Expand the bracket first:

$$6r + 12 = 36$$

Move the $+12$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$$6r = 36 - 12$$

$$6r = 24$$

Move the $\times 6$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$$r = 24 \div 6$$

$$r = 4$$

Example 3: $6 + \frac{k}{5} = -1$

$$6 + \frac{k}{5} = -1$$

Remember, if there is no sign in front it means it is a plus

Move the $+6$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$$\frac{k}{5} = -1 - 6$$

$$\frac{k}{5} = -7$$

Move the $\div 5$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$$k = -7 \times 5$$

$$k = -35$$

Mathematics

Higher

Unit 17

Example 4: $24 - 3m = 6$

$24 - 3m = 6$

Move the $+24$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$-3m = 6 - 24$

$-3m = -18$

Remember, even though it is a – 3, it is being multiplied by the m, so the opposite / inverse operation is a divide

Move the $\times(-3)$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$m = \frac{-18}{-3}$

$m = 6$

Example 5: $7y + 3 = 10y - 6$

$7y + 3 = 10y - 6$

Move the $+3$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$7y = 10y - 6 - 3$

$7y = 10y - 9$

Move the $+10y$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$7y - 10y = -9$

$-3y = -9$

Move the $\times(-3)$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$y = \frac{-9}{-3}$

$y = 3$

Example 6: $5(x - 3) = 4(x + 2)$

$5(x - 3) = 4(x + 2)$

Expand the brackets on both sides

$5x - 15 = 4x + 8$

Move the -15 over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$5x = 4x + 8 + 15$

$5x = 4x + 23$

Move the $+4x$ over the equals sign to the opposite side and change the sign to the opposite / inverse operation

$5x - 4x = 23$

$x = 23$

Mathematics

Higher

Unit 17

Method 2: Balancing equations

Golden Rule: Whatever you do to one side of the equation, you must do exactly the same to the other side to keep the equation in balance

Step 1: If they are not already, get all your unknown letters on one side of the equation (NOT on the bottom of fractions and avoiding negatives).

Step 2: Begin undoing the operations that were done to your unknown letter, by thinking about the order that things were done to the letter

Step 3: Use inverse operations to do this until you are left with just your unknown letter on one side, and the answer on the other

Step 4: Check your answer using substitution to make sure your answer is right.

Example 1: $7p - 3 = 32$

Step 1: The unknown letter (p) only appears on the left-hand side of the equation, there is no negative sign in front of it, and it is not on the bottom of a fraction.

$$7p - 3 = 32$$

Step 2: What order were things done to p? First it was multiplied by the 7, then 3 was subtracted.

Step 3: To undo the operations, we start with the last one, working our way backward and apply the inverse (opposite) operation to both sides:

The last operation was -3, so the opposite / inverse operation is +3, remembering the rule whatever you do to one side of the equation you do to the other.

Now divide both sides by 7

$$7p - 3 = 32$$
$$+3 \qquad +3$$
$$7p = 35$$
$$\div 7 \qquad \div 7$$
$$p = 5$$

Step 4: Check if the answer is right. Substitute $p = 5$ into the initial equation.

When p = 5

$$7p - 3 = 7 \times 5 - 3 = 35 - 3 = 32$$

Example 2: $24 - 3m = 6$

Step 1: The unknown letter (m) only appears on the left-hand side of the equation, it's not on the bottom of a fraction, but it does have a negative sign in front of it.

We can use inverse operations to cancel out the -3m, we just need to add 3m to both sides.

$$24 - 3m = 6$$
$$+3m \qquad +3m$$
$$24 = 6 + 3m$$

Step 2: What order were things done to m? First it was multiplied by the 3, then 6 was added.

Step 3: To undo the operations, we start with the last one, working our way backward and apply the inverse (opposite) operation to both sides:

The last operation was +6, so the opposite / inverse operation is -6, remembering the rule whatever you do to one side of the equation you do to the other.

Now divide both sides by 3

$$-6 \qquad -6$$
$$18 = 3m$$
$$\div 3 \qquad \div 3$$
$$6 = m \quad \text{or} \quad m = 6$$

Step 4: Check if the answer is right. Substitute $m = 6$ into the initial equation.

When m = 6

$$24 - 3m = 24 - 3 \times 6 = 24 - 18 = 6$$

Mathematics

Higher

Unit 17

Example 3: $6 + \frac{k}{5} = -1$

$$6 + \frac{k}{5} = -1$$

$-6 \qquad -6$

$$\frac{k}{5} = -7$$

$\times 5 \qquad \times 5$

$$k = -35$$

Check: Substitute k= -35 into the original equation.

$$6 + \frac{k}{5} = 6 + \frac{-35}{5} = 6 + -7 = 6 - 7 = -1$$

Example 4: $2(3r + 6) = 36$

$$2(3r + 6) = 36$$

Expand brackets

$$6r + 12 = 36$$

$-12 \qquad -12$

$$6r = 24$$

$\div 6 \qquad \div 6$

$$r = 4$$

Check: Substitute $r = 4$ into the original equation.

$$2(3r + 6) = 2(3 \times 4 + 6) = 2(12 + 6) = 2 \times 18 = 36$$

Example 5: $7y + 3 = 10y - 6$

$$7y + 3 = 10y - 6$$

$-7y \qquad -7y$

$$3 = 3y - 6$$

$+6 \qquad +6$

$$9 = 3y$$

$\div 3 \qquad \div 3$

$3 = y$ or $y = 3$

Check: $10y - 6 = 10 \times 3 - 6 = 24$

Example 6: $5(x - 3) = 4(x + 2)$

$$5(x - 3) = 4(x + 2)$$

expand $\qquad$ expand

$$5x - 15 = 4x + 8$$

$-4x \qquad -4x$

$$x - 15 = 8$$

$+15 \qquad +15$

$$x = 23$$

Check: Substitute $c = 23$ into the original equation.

$$5(23 - 3) = 4(23 + 2)$$

$$5 \times 20 = 4 \times 25$$

$$100 = 100$$

Mathematics

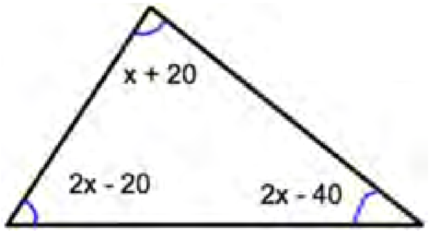
Higher

Unit 17

Forming and Solving Equations

Sometimes we are given information and need to form an equation using the information, before solving the equation.

Example 1: The angles in the triangle are $(x + 20)°$, $(2x - 20)°$, and $(2x - 40)°$. Form an equation and use it to find the value of x.

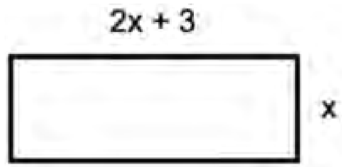


Angles in a triangle add to 180°, so $(x + 20)$ plus $(2x - 20)$ plus $(2x - 40)$ is equal to 180.

$$x + 20 + 2x - 20 + 2x - 40 = 180$$
$$5x - 40 = 180$$
$$5x = 220$$
$$x = 44°$$

Example 2: The perimeter of the rectangle is 42cm.

a) Form an equation in x and solve it to find the value of x.



The perimeter is the distance all the way around the shape, so a length $(2x + 3)$ plus a width (x) plus another length $(2x + 3)$ plus another width (x) is equal to 42cm.

$$2x + 3 + x + 2x + 3 + x = 42$$
$$6x + 6 = 42$$
$$6x = 36$$
$$x = 6cm$$

b) Calculate the area of the rectangle.

The length of the rectangle is $2 \times 6 + 3 = 15cm$

The width of the rectangle is $6cm$

So, the area of the rectangle is $15 \times 6 = 90cm^2$

Example 3: Jane is 4 years older than Tom.

David is twice as old as Jane.

The sum of their three ages is 60.

Form an equation and use it to find the age of each person.

Let Tom's age $= x$

Jane's age $= x + 4$

David's age is $2(x + 4) = 2x + 8$

The sum of their ages is 60:

$$x + x + 4 + 2x + 8 = 60$$
$$4x + 12 = 60$$
$$4x = 48$$
$$x = 12$$

So, Tom is 12 years old

Jane is $12 + 4 = 16$ years old

David is $2 x 12 + 8 = 32$ years old.

Mathematics

Higher

Unit 17

Solving Fractional Linear Equations

Method 1 – Using the LCM of the Denominators

Skill check 1: I can find the LCM (Lowest Common Multiple) from a set of numbers

The LCM, is the lowest number that is in the times table of your numbers

e.g. What is the LCM of 2, 5 and 10?

Multiples of 2: 2 4 6 8 **10** 12…
Multiples of 5: 5 **10** 15 20…
Multiples of 10: **10** 20 30…
The LCM of 2, 5 and 10 is **10**

Skill check 2: I can solve linear equations

e.g. Solve $7(2x-5)=-42$

Expand brackets $14x-35=-42$

Collect like terms $14x=-7$

x by itself (÷ 14) $x=\frac{-7}{14}$

$x=-\frac{1}{2}$ or -0.5

Or $7(2x-5)=-42$

Brackets by self (÷ 7) $2x-5=-6$

Collect like terms $2x=-1$

x by itself (÷ 2) $x=-\frac{1}{2}$ or -0.5

Examples:

Step 1: Find the LCM of **all** the denominators (bottom number on fraction).
In example 2 the LCM of 6, 3 and 2 is 6.

Step 2. Bracket each numerator (top number in fraction) and multiply by the LCM. Cancel out the denominators against the numerator. Multiply out any brackets.

Step 3: Collect like terms and solve for x. You may use the 'balance' method as in example 1 or the 'change side, change sign' method as in example 2.

Example 1: Solve the following

$$\frac{2x-1}{5}-\frac{6x+3}{4}=1$$

$$\frac{20(2x-1)}{5}-\frac{20(6x+3)}{4}=-1$$

$$\frac{{}^{4}\cancel{20}(2x-1)}{\cancel{5}}-\frac{{}^{5}\cancel{20}(6x+3)}{\cancel{4}}=-1$$

$$8x-4-30x-15=-20$$

$$-22x-19=-20$$

$+19$ $+19$

$$-22x=-1$$

$\div(-22)$ $\div(-22)$

$$x=\frac{-1}{-22}$$

$$=\frac{1}{22}$$

Example 2: Solve the following

$$\frac{2x-3}{6}-\frac{x+2}{3}=\frac{5}{2}$$

$$\frac{6(2x-3)}{6}+\frac{6(x+2)}{3}=\frac{6(5)}{2}$$

$$\frac{{}^{1}\cancel{6}(2x-3)}{\cancel{6}}+\frac{{}^{2}\cancel{6}(x+2)}{\cancel{3}}=\frac{{}^{3}\cancel{6}(5)}{\cancel{2}}$$

$$2x-3+2x+4=15$$

$$4x+1=15$$

$$4x=15-1$$

$$4x=14$$

$$x=\frac{14}{4}$$

$$=3\frac{1}{2}$$

Mathematics

Higher

Unit 17

Method 2 – Using a Common Denominator

Skill check 1: I can add and subtract fractions by making sure the fractions have the same denominator

e.g. $\frac{5}{9}+\frac{3}{7}=\frac{35}{63}+\frac{27}{63}=\frac{62}{63}$

Skill check 2: I can solve linear equations (see previous page)

To solve linear fractional equations with this method, you must make sure the denominators are the same. Remember whatever you do to the bottom of the fraction (denominator), you must do to the top (numerator).

Examples:	**Example 1:** Solve the following	**Example 2:** Solve the following
Step 1: Make the denominators of each fraction the same. You may have to expand out brackets to tidy up. Step 2: As you now have the same denominators you can add/subtract the fractions. Simplify if possible. Step 3: Solve the equation using the 'balance' method (example 1) or 'change side, change signs' method (example 2).	$\frac{x}{2}+\frac{3x}{5}=22$ $\frac{5x}{10}+\frac{6x}{10}=22$ $\frac{5x+6x}{10}=22$ $\frac{11x}{10}=22$ x10 x10 $11x = 220$ ÷11 ÷11 $x = 20$	$\frac{x+3}{3}-\frac{x-4}{5}=3$ $\frac{5(x+3)-3(x-4)}{15}=3$ $\frac{5x+15-3x+12}{15}=3$ $\frac{2x+27}{15}=3$ $2x+27=3\times 15$ $2x+27=45$ $2x=45-27$ $2x=18$ $x=9$

Mathematics

Higher

Unit 18

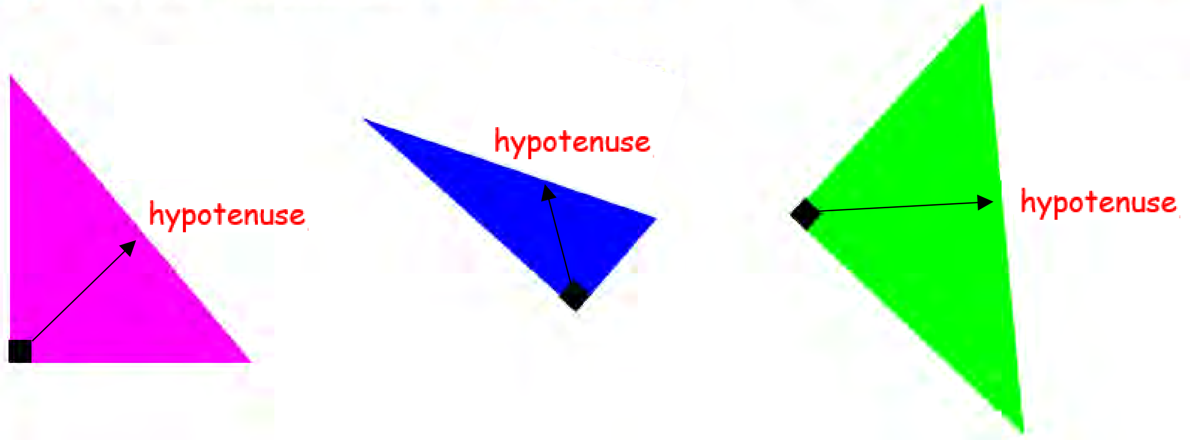
Pythagoras' Theorem

We use Pythagoras' Theorem if we have a right-angled triangle, are given the lengths of two sides, and are asked to find the length of the other side.

Pythagoras' Theorem can be written in two ways depending on which side of the triangle you need to find:
the longest side, called the hypotenuse,
or one of the other two shorter sides.

The two ways are simply different arrangements of the same original formula, so, if you are good at formula re-arranging, then you only need to remember one.

The Hypotenuse is the longest side of the right-angled triangle, and it is the side opposite the right-angle. The other two sides are the shorter sides.



Note: Sometimes you will need to create the right-angled triangle.

Mathematics

Higher

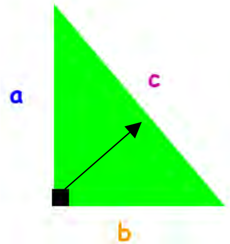
Unit 18

Finding the Length of the Hypotenuse (the Longest Side)

1. Label the hypotenuse **c**, and the other sides **a** and **b**

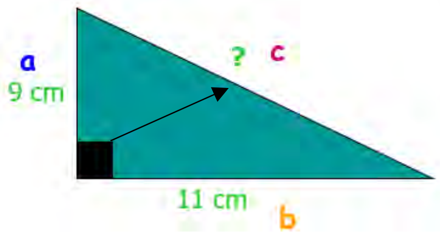
2. Use the following formulae:

$$c^2 = a^2 + b^2$$



3. Replace the letters with the numbers you have been given, and solve.

Example:



The side we are looking for is the hypotenuse.

Step 1. Label the sides

Step 2. Use the formula: $c^2 = a^2 + b^2$

Step 3. Put in the numbers:

$$c^2 = 9^2 + 11^2$$
$$c^2 = 81 + 121$$
$$c^2 = 202$$
$$c = \sqrt{202}$$
$$c = 14.2cm \quad (1\text{dp})$$

Square root both sides!

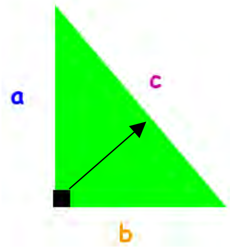
Check: The hypotenuse is the longest side of the triangle, 14.2cm is longer than the other two sides.

Finding the Length of a Shorter Side

1. Label the hypotenuse **c**, label the side you want to find **a**, and the other side **b**

2. Use the following formulae:

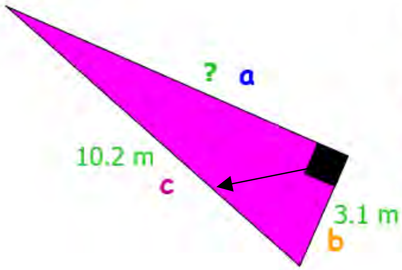
$$a^2 = c^2 - b^2$$



3. Replace the letters with the numbers you have been given, and solve.

Note: As mentioned before, this is just a different arrangement of: $c^2 = a^2 + b^2$

Example:



The side we are looking for is one of the shorter sides

Step 1. Label the sides

Step 2. Use the formula: $a^2 = c^2 - b^2$

Step 3. Put in the numbers:

$$a^2 = 10.2^2 - 3.1^2$$
$$a^2 = 104.04^2 - 9.61^2$$
$$a^2 = 94.43$$
$$a = \sqrt{94.43}$$
$$a = 9.72m \quad (2\text{dp})$$

Square root both sides!

Check: a = 9.72m which is shorter than the hypotenuse (the longest side).

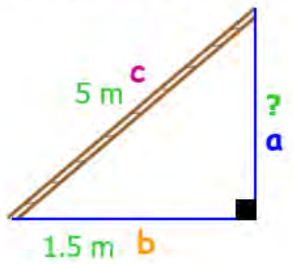
Mathematics

Higher

Unit 18

Further Examples:

A 5m ladder rests against the side of a house. The foot of the ladder is 1.5m away from the house. How far up the side of the house does the ladder reach?



Step 1. Draw a picture using the information in the question.

Step 2. Label the sides

Step 3. We are looking for a shorter side so use the formula: $a^2 = c^2 - b^2$

Step 4. Put in the numbers:

$$a^2 = 5^2 - 1.5^2$$

$$a^2 = 25^2 - 2.25^2$$

$$a^2 = 22.75$$

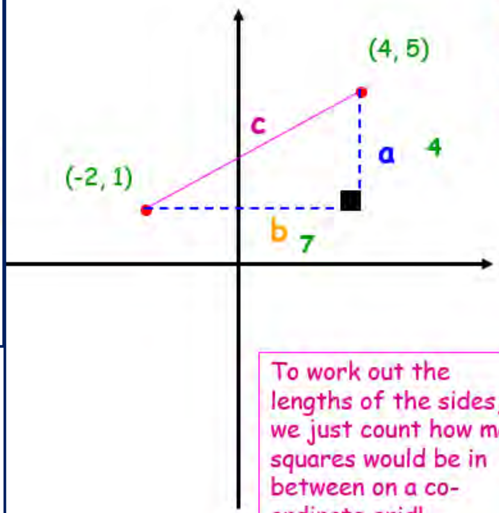
$$a = \sqrt{22.75}$$

$$a = 4.77m \quad (2dp)$$

Square root both sides!

Find the distance between these two co-ordinates: (4, 5) and (-2, 1)

Again, this does not look like a Pythagoras question, but if you plot the coordinates it looks more obvious.



To work out the lengths of the sides, we just count how many squares would be in between on a co-ordinate grid!

Step 1. Draw a picture using the information in the question.

Step 2. Label the sides

Step 3. We are looking for the hypotenuse (the longest side) so use: $c^2 = a^2 + b^2$

Step 4. Put in the numbers:

$$c^2 = 4^2 + 7^2$$

$$c^2 = 16 + 49$$

$$c^2 = 65$$

$$c = \sqrt{65}$$

$$c = 8.1 \quad (1dp)$$

Square root both sides!

Mathematics

Higher

Unit 18

Proving Right-Angled Triangles

If a triangle **does not** have the square in the corner that indicates it is 90° **and** we are only given the lengths of the three sides, then we can **use Pythagoras' Theorem to prove if a triangle is a right-angled triangle** or if it is not a right-angled triangle.

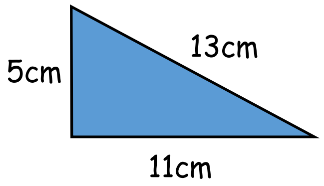
Method:

Step 1: Square the 2 shorter sides and add them together.

Step 2: Square the longest side (if the triangle is right-angled this would be the hypotenuse).

Step 3: If the answer to step 1 and the answer to step 2 are equal, the triangle is right-angled, if the answers are not equal, the triangle is not right-angled.

Example 1: Is it possible to draw a right-angled triangle with the measurements shown below?



Step 1: Square the shorter sides and add together.

$$5^2 + 11^2 = 25 + 121$$
$$= 146$$

Step 2: Square the longest side.

$$13^2 = 169$$

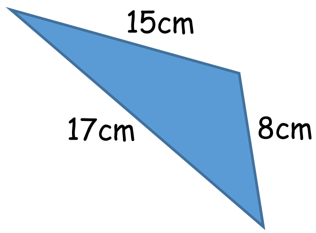
Not equal to

Step 3: $146 \neq 169$

The triangle <u>is not</u> a right-angled triangle

Note: We cannot just assume a triangle is right-angled because it looks like it is, or that a triangle is not right-angled because it does not look like it is.
We must use Pythagoras' Theorem to check.

Example 2: Is the triangle below a right-angled triangle?



Step 1: Square the shorter sides and add together.

$$8^2 + 15^2 = 64 + 225$$
$$= 289$$

Step 2: Square the longest side.

$$17^2 = 289$$

Step 3: $289 = 289$

The triangle <u>is</u> a right-angled triangle

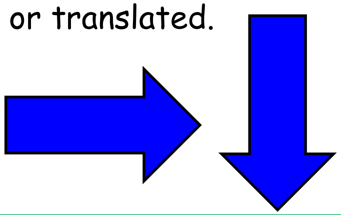
Mathematics

Higher

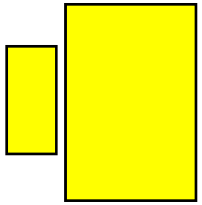
Unit 19

Congruent shapes

Congruent shapes have the same shape and size, but could be rotated, reflected, or translated.

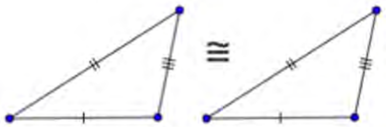
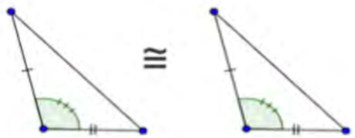
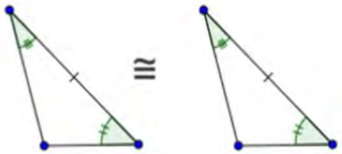


The blue arrows are congruent, they are the same shape and size; one has been rotated.

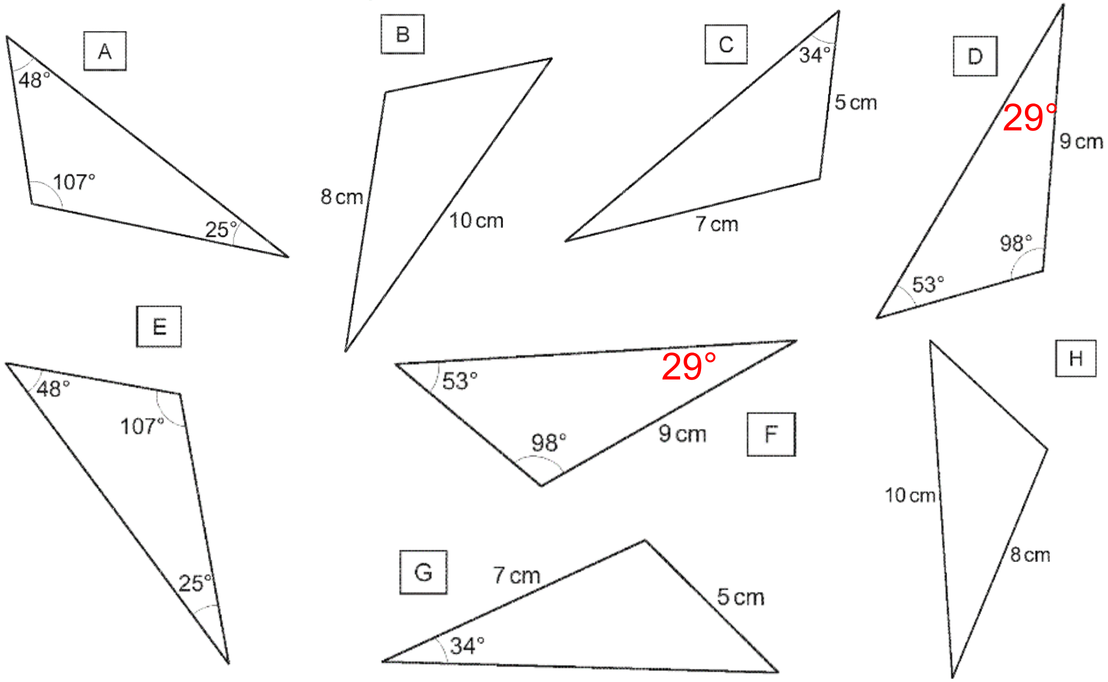


The yellow rectangles are not congruent, they are the same shape but not the same size.

There are **four rules of congruency** for triangles that prove whether a triangle is congruent or not.

	SSS (side, side, side): 3 sides on one triangle are equal to those on another triangle.
	SAS (side, angle, side): 2 sides with the included angle on one triangle are equal to those on another triangle.
	ASA (angle, side, angle): 2 angles with the included side on one triangle are equal to those on another triangle.
	RHS (right-angle, hypotenuse, side): The hypotenuse and another side on one triangle are equal to those on another triangle.

Example: The eight triangles below have not been drawn to scale.
Some information about the lengths of the sides or the sizes of the internal angles have been included on each diagram.



Using only the information given, state which 2 triangles are congruent and give the condition of congruency.

The congruent triangles are ....D.... and ....F.... .

The condition of congruency is: ASA – 2 angles with the included side

Mathematics

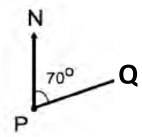
Higher

Unit 20

Bearings and Scale Drawings

Scales are used to reduce real world dimensions to a useable size.

A **bearing** is an angle, measured from the **north** line in a **clockwise** direction. It is given as a **3-digit** number.



The bearing of Q from P is given as 070°. The easiest way of thinking about it is you are standing at P, facing north what angle would you need to turn to face Q.

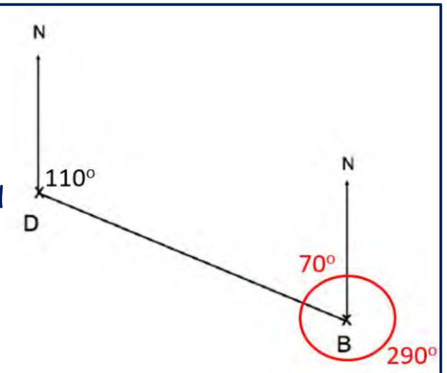
You may need to draw the north line directly upwards before constructing a bearing.

For bearing questions where diagrams are drawn to scale, you will need to use a protractor to measure or draw angles and use a ruler for measurements.

For bearing questions where diagrams are not drawn to scale, you will need to recall certain facts about angles, such as

- Angles on a straight line add up to 180°.
- Angles around a point add up to 360°.
- Interior angles add up 180°.
- Alternate angles are equal.
- Corresponding angles are equal.

Example 1:
The diagram shows the position of a boat B and dock D. The scale of the diagram is 1cm to 5km.



a) Calculate the real distance between the boat and the dock.

The length from D to B measures 4.5cm

$4.5 \times 5 = 22.5km$

b) State the bearing of the boat from the dock.

110^o (from D to B)

c) Calculate the bearing of the dock from the boat.

$180^o - 110^o = 70^o$ (the angles are interior)

$360^o - 70^o = 290^o$ (angles around a point equal 360°)

Mathematics

Higher

Unit 20

Example 2

The diagram is a sketch of Swansea bay with the positions of Mumbles, Swansea and Porthcawl marked.

a) By drawing and measuring an angle, find the bearing of Swansea from Porthcawl.

First, join up Swansea and Porthcawl with a straight line.

Remember bearings are measured from the north line in a clockwise direction, from Porthcawl this is a reflex angle. A normal protractor only goes up to 180°.

Therefore, the easiest thing to do is measure the acute angle anticlockwise from the north line and subtract from 360°. Using a protractor this acute angle measures 35°.

$$360° - 35° = 325°$$

So, the bearing of Swansea from Porthcawl is 325°

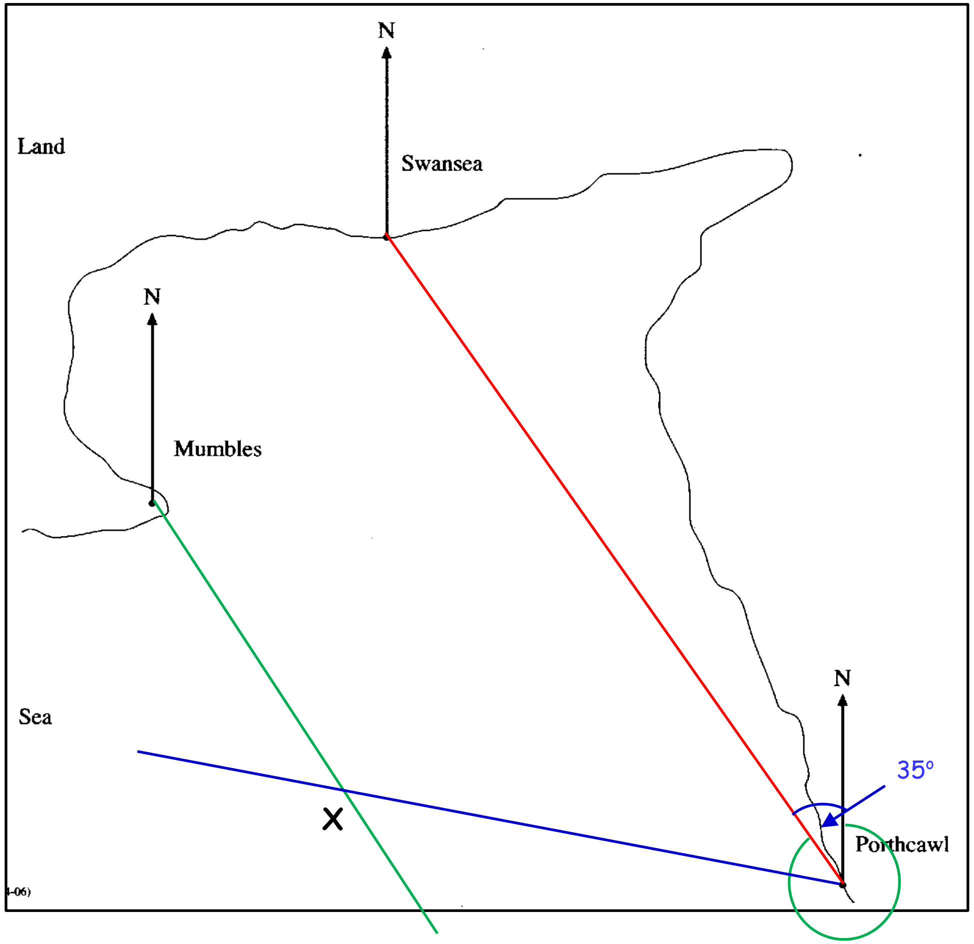
b) A ship is on a bearing of 145° from Mumbles and on a bearing of 283° from Porthcawl. Draw these bearings and mark the position of the ship X.

Measure 145° from Mumbles and draw a straight line going through the angle.

You cannot draw an angle of 283° with a normal protractor so subtract from 360° and draw the acute angle anticlockwise.

$$360° - 283° = 77°$$

Draw a straight line through this angle and where the two lines intersect (cross) must be the position of the ship X.



Mathematics

Higher

Unit 21

Simultaneous Equations

Simultaneous Equations are two equations, each containing two unknown letters, and you must use both equations to find the value of your unknown letters.

Key Point: The values you find for your unknown letters must make BOTH equations balance – you can check your answers and make sure that you have got it right.

Method for Solving Simultaneous Equations:

Step 1: If you need to, re-arrange your equations so they are in the same form

Step 2: Write one equation underneath the other, lining up the unknown letters

Step 3: Choose one of the unknown letters and use your algebra skills to change one or both of the equations to make sure there are the same number (don't worry about sign) of your chosen letter in each equation. Your chosen letter becomes your Key Letter.

Step 4: Put a box around your Key Letters and their sign

Step 5: Follow this rule:

If the signs are the same, subtract the two equations

If the signs are different, then add the two equations

Step 6: If you have done this correctly, your Key Letter should cancel out and you should be left with just one equation with one unknown

Step 7: Solve this equation to work out the value of the unknown letter

Step 8: Choose one of the original equations and substitute in the answer you found in **Step 7.** to work out the value of the other letter.

Step 9: Check your answers are correct using the equation you **did not** choose in **Step 8.**

Example 1: Solve $3x + y = 19$ and $x + y = 9$

1. The equations are in the same form, some x's and some y's, equal a number.

2. Write the second equation underneath the first.

$$3x + y = 19 \quad (1)$$
$$x + y = 9 \quad (2)$$

3. There are already the same number of y's in both equations (there is an invisible 1 in front of both), so let us choose the y's to be our Key Letters.

4. Put a box around our Key Letters, and their signs.

(1) – (2)

$$\begin{array}{r} 3x \boxed{+y} = 19 \\ x \boxed{+y} = 9 \\ \hline 2x \quad = 10 \end{array} \; -$$

5. The signs of our Key Letters are the same (both +) so we must subtract equation (2) from equation (1)

6. Our Key Letters have cancelled out.

7. Solve the equation.

$$2x = 10$$
$$x = \frac{10}{2}$$
$$x = 5$$

8. Substitute this value in one of the original equations to find the value of the other unknown letter.

Substitute in (2) :

$$5 + y = 9$$
$$y = 9 - 5$$
$$y = 4$$

9. We now have our two solutions. Check them using the equation we did not choose in 8.

Check in (1) :

$$3 \times 5 + 4 = 9$$

The solutions are, $x = 5$ and $y = 4$

Mathematics

Higher

Unit 21

Example 2: $3x - 2y = 3$ and $2x + 2y = 12$

$$3x \boxed{-2y} = 3 \quad (1)$$
$$2x \boxed{+2y} = 12 \quad (2)$$

Signs of Key Letters are different so add

$$3x - 2y = 3$$
$$2x + 2y = 12 \quad +$$
$$5x \quad = 15$$
$$x = \frac{15}{5}$$
$$x = 3$$

Substitute in (2) :

$$2 \times 3 + 2y = 12$$
$$6 + 2y = 12$$
$$2y = 12 - 6$$
$$2y = 6$$
$$y = \frac{6}{2}$$
$$y = 3$$

Check in (1) :

$$3 \times 3 - 2 \times 3 = 3$$

Solutions are $x = 3$ and $y = 3$

Example 3: $2x + 3y = 7$ and $3x + 5y = 18$

$$2x + 3y = 7 \quad (1)$$
$$3x + 5y = 18 \quad (2)$$

Multiply (1) by 3: $\boxed{6x} + 9y = 21$

Multiply (2) by 2: $\boxed{6x} + 10y = 36$

Signs of Key Letters are the same so subtract

$$6x + 10y = 36$$
$$6x + 9y = 21 \quad -$$
$$y = 15$$

Substitute in (1) : $2x + 3 \times 15 = 7$

$$2x + 45 = 7$$
$$2x = 7 - 45$$
$$2x = -38$$
$$x = -\frac{38}{2}$$
$$x = -19$$

Check in (2) :

$$3 \times (-19) + 5 \times 15 = 18$$

Solutions are $x = -19$ and $y = 15$

Example 4: $7x - 2y = -20$ and $3x = 6 - 4y$

Rearranging the second equation gives: $3x + 4y = 6$

$$7x - 2y = -20 \quad (1)$$
$$3x + 4y = 6 \quad (2)$$

Multiply (1) by 2: $14x - 4y = -40$

Signs of Key Letters are different so add:

$$14x \boxed{-4y} = -40$$
$$3x \boxed{+4y} = 6 \quad +$$
$$17x \quad = -34$$
$$x = -\frac{34}{17}$$
$$x = -2$$

Substitute in (2) : $3 \times (-2) + 4y = 6$

$$-6 + 4y = 6$$
$$4y = 6 + 6$$
$$4y = 12$$
$$y = \frac{12}{4}$$
$$y = 3$$

Check in (1) :

$$7 \times (-2) - 2 \times 3 = -20$$

Solutions are $x = -2$ and $y = 3$

Mathematics

Higher

Unit 21

Forming and Solving Simultaneous Equations

Sometimes we are given information and need to form the simultaneous equations from the information before solving them.

Example: Simon and Syra are on holiday in Devon. They buy some holiday souvenirs for their friends. Simon pays £2.05 for 2 key rings and 3 pencils. Syra pays £3.20 for 3 key rings and 5 pencils. All the key rings are the same price and all the pencils are the same price.

Find the individual prices of a key ring and a pencil. You must use an algebraic method.

Find the price of one key ring, find the price of one pencil

Use algebra to solve, in this case simultaneous equations

Step 1: Form the simultaneous equations

Let x represent a key ring, let y represent a pencil

x and y are the most common letters used but you can use any letters (we could have chosen k to represent keyrings and p to represent pencils).

Simon's purchases would be: $2x + 3y = 2.05$ — 2 keyrings = $2x$, 3 pencils = $3y$, total price = £2.05.

Syra's purchases would be: $3x + 5y = 3.20$ — 3 keyrings = $3x$, 5 pencils = $5y$, total price = £3.20.

We now have our two equations and can solve like the previous examples.

Step 2: Solve the simultaneous equations

$2x + 3y = 2.05$ (1)

$3x + 5y = 3.20$ (2)

Multiply (1) by 3: $6x + 9y = 6.15$

Multiply (2) by 2: $6x + 10y = 6.40$

Signs of Key Letters are the same so subtract

$6x + 10y = 6.40$

$6x + 9y = 6.15$ −

$y = 0.25$

Substitute in (1) : $2x + 3 \times 0.25 = 2.05$

$2x + 0.75 = 2.05$

$2x = 2.05 - 0.75$

$2x = 1.30$

$x = \frac{1.30}{2}$

$x = 0.65$

Check in (2) :

$3 \times 0.65 + 5 \times 0.25 = 3.20$

Solutions are $x = 0.65$ and $y = 0.25$

Step 3: Answer the question

A key ring, x, costs £0.65 or 65p, a pencil, y, costs £0.25 or 25p.

Note: The question could ask you to find the cost of a certain number of keyrings and pencils. For example, find the cost of 4 keyrings and 2 pencils. You would then use substitution to work this out.

$4 \times 0.65 + 2 \times 0.25 = 3.1$ So, 4 keyrings and 2 pencils would cost £3.10

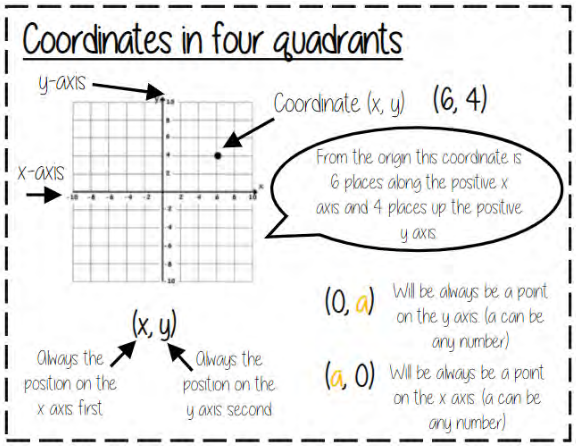
Mathematics

Higher

Unit 22

Straight Line Graphs

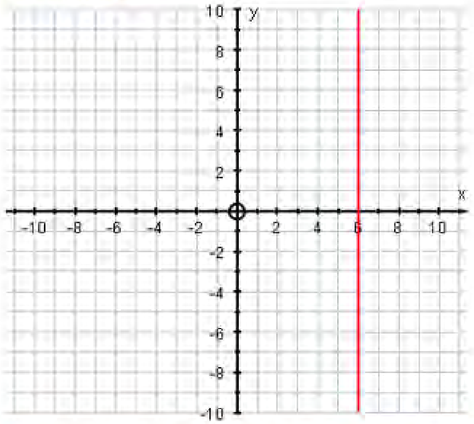
Recap: How to plot coordinates:



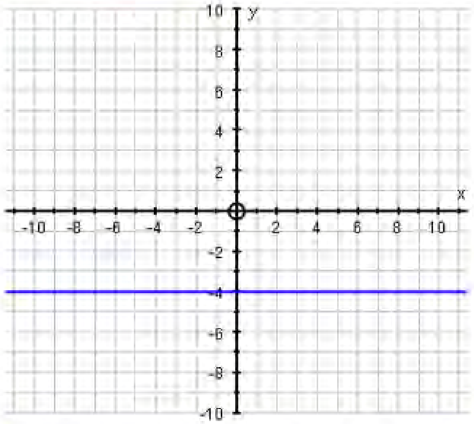
Graphs of $x = ?$ and $y = ?$

You need to learn how to recognise and draw horizontal and vertical lines.

Examples:



Every single point on this line has an x co-ordinate of 6, so the equation of the line is: **x = 6**



Every single point on this line has a y co-ordinate of -4, so the equation of the line is: **y = -4**

Note: The equation of the x axis is y = 0, and the equation of the y axis is x = 0.

What Does the Equation of a Straight Line Actually Mean?

The equation of a straight line is just a way of writing the relationship between the x coordinates and the y coordinates that lie on that line.

Example: $y = 2x - 1$

This says that the relationship between all the x coordinates and all the y coordinates is "take the x coordinate, multiply it by 2, subtract 1, this gives the y coordinate".

So, if you had these coordinates $(5, 9)$ then it is on the line (5 x 2 – 1 = 9 which is the y coordinate), but if you had the coordinates $(3, 2)$ then it is not on the line (3 x 2 – 1 = 5 which is not the y coordinate).

You end up with a straight line that goes through all the coordinates which share that relationship.

Mathematics

Higher

Unit 22

Drawing Straight Line Graphs from their Equation

As well as graphs of horizontal and vertical lines, there are also graphs of diagonal lines.

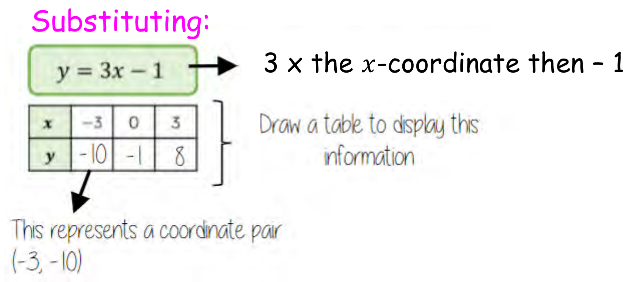
Method for Drawing Straight-Line Graphs

1. If the question does not give you values of x to use, then choose sensible values of x (A good choice of x values are 0, 1 and 2. This will show you the direction of the line. You need **at least** 3 values of x but choosing 4 (values -1, 0, 1 and 2) would make it even better).

2. Carefully substitute each x value it into the equation to get your y values, be careful if substituting negative numbers.

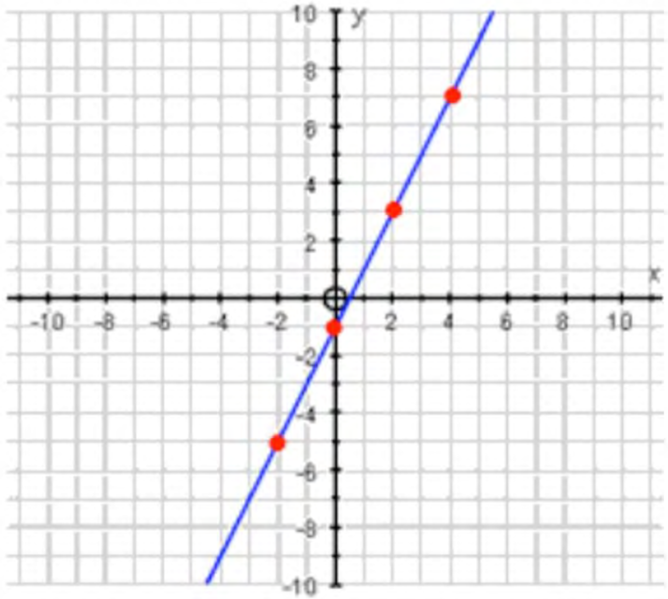
3. Join up the points with a straight line

Note: The points should make a straight line, if one of your points does not lie on the straight line, check your substitution again.



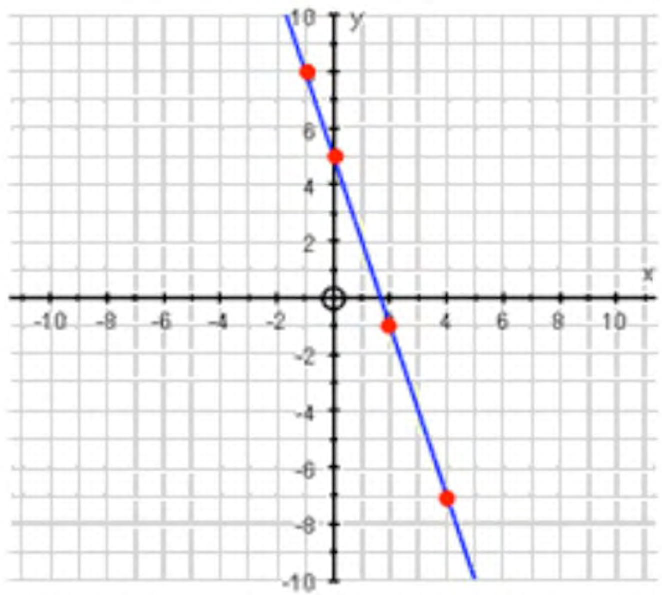
$y = 2x - 1$

x	0	2	4	-2
y	-1	3	7	-5



$y = -3x + 5$

x	0	2	4	-1
y	5	-1	-7	8



Mathematics

Higher

Unit 22

The Equation of a Line in the Form $y = mx + c$

$$y = mx + c$$

The gradient m

- This tells you the gradient/steepness of the line
- The bigger the number, the steeper the line
- If the number is positive, the line slopes upwards
- If it is negative, the line slopes downwards
- Parallel lines have the same gradient

Example: $y = 4x + 7$ has a gradient of 4, $y = -9x + 3$ has a gradient of -9.

The y-intercept c

- This tells you where the line crosses the y axis

Example: $y = 2x + 5$ crosses the y-axis at 5, $y = 2x - 5$ crosses the y-axis at -5.

Note: If the equation is NOT in the form: y = mx + c, you must first re-arrange it.

If you are given points and not a line all you need to do is join the points up to give you a line before you start.

Note: You may be asked to draw a line given an equation which looks like this:

$$9x + 3y = 18.$$

Fist, rearrange the equation into the form $y = mx + c$.

$$9x + 3y = 18$$

$$3y = 18 - 9x$$

$$y = \frac{18}{3} - \frac{9x}{3}$$

$$y = 6 - 3x$$

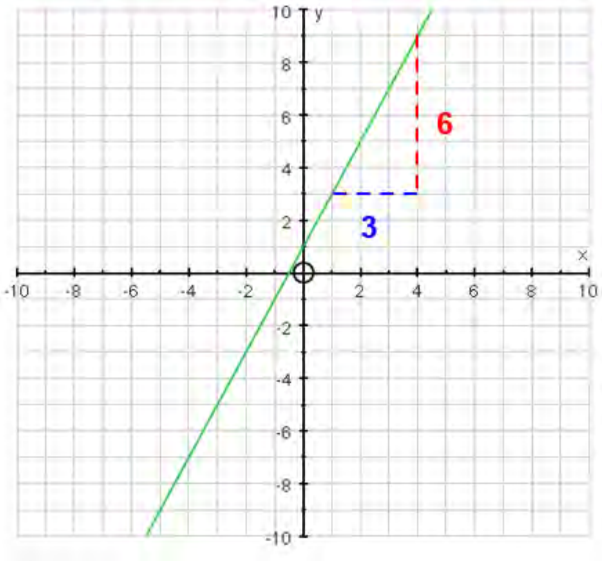
$$y = -3x + 6$$

$$y = mx + c.$$

Then, follow the steps as above.

Working Out the Equation of a Line

Using our knowledge of y = mx + c, we can actually work backwards and figure out the equation of a straight line by its graph.



First, we must work out the gradient of the line by drawing a right-angled triangle anywhere on the line and using this formula:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x}$$

$$\text{Gradient} = \frac{6}{3} = 2$$

Then we work out the y-intercept, which is the place the line crosses the y axis, (0, 1)

So, the equation of the line is:

$$y = 2x + 1$$

Mathematics

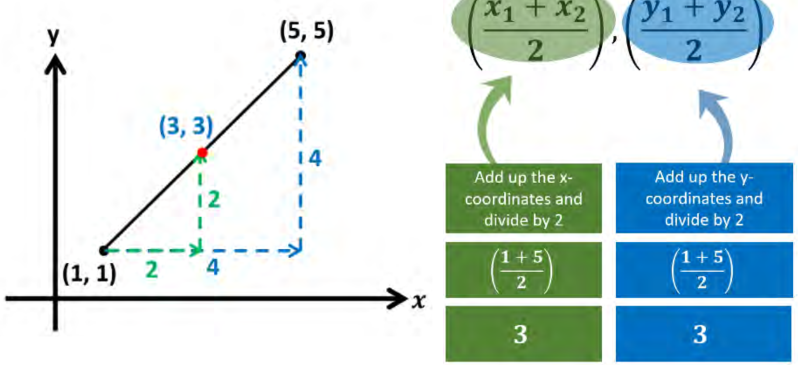
Higher

Unit 22

Midpoint of a Line

The midpoint is exactly half-way between two coordinates

The Midpoint of a Line



Equations of Parallel and Perpendicular Lines

Parallel lines *are lines that never meet* – they are always a fixed distance apart. Lines that are parallel will have the same **gradient**.

i.e. The graphs of $y = 2x + 1$ and $y = 2x - 2$ have the same gradient of 2 so they are **parallel**.

Example

Write an equation of a line that is parallel to $y = 7x + 5$

The gradient of the line in the question is **7**. So, any line with a gradient of 7 will be parallel.

Two examples of answers you could give are: $y = 7x - 15$ and $y = 7x + 3.5$

Two lines are ***perpendicular*** *if they meet at a right angle.* Two lines are perpendicular if the product of their gradient is -1.

Example

Find the equation of a straight line that is perpendicular to $y = 3x + 2$

The gradient of $y = 3x + 2$ is 3.

To find the perpendicular gradient you need to find the number which multiplies by 3 to give -1. This is called the negative reciprocal.

The negative reciprocal of 3 is $-\frac{1}{3}$ So, any line with the gradient $-\frac{1}{3}$ will be perpendicular.

i.e. $y = -\frac{1}{3}x + 5$ or $y = -\frac{1}{3}x - 6$

Using Straight Line Graphs to Solve Simultaneous Equations

It is possible to use straight line graphs to solve simultaneous equations

All you need to do it to carefully plot both lines, and the point where they cross is the solution, but remember you want $x =$ and $y =$

Example: Solve the following pair of simultaneous equations graphically:

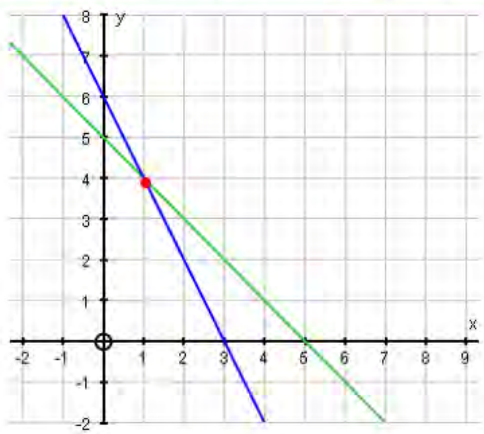
$x + y = 5$ and $2x + y = 6$

$x + y = 5$

x	0	5
y	5	0

$2x + y = 6$

x	0	3
y	6	0



The solution is $x = 1$ and $y = 4$

Mathematics

Higher

Unit 23

Averages and Representing Data

The main averages, which can also be referred to as **measures of central tendency**, are the mean, mode and median. **Central tendency** is a single value that attempts to describe a set of data by identifying the central position within that set of data.

The mean	The mode	The median
The mean uses all the values in the data. To calculate the mean: 1. Add up all of the items 2. Divide by how many items there are	The mode is the most common value that appears in the data and there can be more than one. If there are two modes, we say it is **bimodal**; if there are more than two modes it is **multimodal**. If all the values appear the same number of times, then **there is no mode**. Ordering the numbers is helpful.	The median is the middle value in the sorted set of data. To calculate the median: 1. List the values in order from smallest to largest (ascending order) 2. Cross values off from each end to identify the middle value If there are two numbers in the middle, you must calculate the mean of these two values. This means we add them up and divide by 2.

Example 1: Find the mean, mode and median of the following set of numbers 10, 2, 3, 5, 15, 19, 21, 5

Mean = $\frac{2+3+5+5+10+15+19+21}{8} = \frac{80}{8} = 10$

Mode = 5

Median ~~2~~, ~~3~~, ~~5~~, 5, 10, ~~15~~, ~~19~~, ~~21~~ $\frac{5+10}{2} = 7.5$

Example 2: Finding the total when given the mean of a set of numbers

The mean of a set of 6 numbers is 5. What is the total of the 5 numbers?

Remember, to find the mean $\frac{total\ value\ of\ items}{number\ of\ items} = mean$

Therefore, to find the original total of the numbers we use

$$total\ value\ of\ items = mean \times number\ of\ items$$

$total\ value\ of\ items = 5 \times 6 = 30$

The range

The range is not an average but a measurement of **spread of data**. The smaller the range the more consistent the data. The range is found by calculating the difference between the highest and lowest value. The range of the data in example 1 is 21 – 2 = 19.

Mathematics

Higher

Unit 23

Real Life Use

The mean, median and mode are all valid measures of central tendency, but in certain situations, some are more appropriate to use than others. You must be able to select and use the appropriate measure and/or range to compare two distributions in real life practical situations.

	Mode	**Median**	**Mean**
Advantages	- Very easy to find - Not affected by outliers (extreme values) - Can be used for non-numerical data like category's (e.g. car colour)	- Easy to find for ungrouped data - Less affected by outliers and skewed data (See unit 31 for more information on outliers and skewed data)	- Easy to find but often has to be calculated - Uses all the values - The total for a given number of values can be calculated from it (see example 2)
Disadvantages	- Doesn't use all the values - May have multiple 'modes' - May not exist	-Doesn't use all the values	- Outliers can distort it
Best used for	- Qualitative data (data you describe like car colour) - Finding the most likely value	- Quantitative data (numerical data) - Data with outliers	-Quantitative data (numerical data) - Data whose values are spread in a balanced way (no outliers)

Example 3

A company has placed an advertisement online as seen on the right.
In reality, one Manager earns £120,000 per year and nine others earn £20,000 per year.

a) What average have they used in the advert? They have used the mean.

b) Was this a suitable average to use? If not, what would be a suitable average to use?
No, it is not suitable. Although it took into account all the pieces of data, the data includes a possible outlier of £120,000. £30,000 does not reflect what you would probably get paid in the firm. The median or mode is more relevant in this situation as you would probably earn £20,000.

c) Why did they use this average? It is a correct 'average' for the data and will attract more job candidates as it is higher than the other averages.

Mathematics

Higher

Unit 23

Different Types of Data

Discrete data is data that can only take on certain values, like the number of students in a class (you cannot have half a student) or shoe size (you can have size 5 or 5.5 but not 5.67).

Continuous data is data that can take on any value, like age, height, weight, temperature, or length are other examples of continuous data.

You can think of is as, Continuous data is measured, and Discrete data is counted.

Finding the Mean, Median and Mode from a Frequency Table

Example: A team plays 20 games; the coach records the number of goals they score in each game in a frequency table.

Number of Goals	Frequency	fx
0	5	0 x 5 = 0
1	6	1 x 6 = 6
2	4	2 x 4 = 8
3	3	3 x 3 = 9
4	2	4 x 2 = 8
	$Total\ f = 20$	$Total\ fx = 20$

Mean goals = $\frac{Total\ fx}{Total\ f} = \frac{31}{20} = 1.55$

Mean: To find the mean, you need to find the total value of all the data. Then divide by the total frequency as seen to the left.

Median: To find the median, we need to work out what position in the data the median will be. If there are n pieces of data, the median value will be in position $\frac{n+1}{2}$. In this case the median position is $\frac{20+1}{2}$ =10.5th

The first row covers the first 5 positions so the 10.5th position would be in the second row; therefore, the Median is 1 goal.

Mode: The mode is the group that contains the highest frequency. Therefore, the mode is 1 goal.

Mathematics

Higher

Unit 23

Estimating the Mean, Finding the Median and Modal Class from a Grouped Frequency Table

In the previous set of notes, we found the exact mean, median and mode of the dataset. Unfortunately, we cannot calculate the exact mode, median or mean of grouped data as we do not know the individual items data. We can, however, make some comments on the data.

Estimating the mean

The frequency table shows pupil ages. Find the mean.

Age	Frequency
8 - 10	12
11 - 13	25
14 - 16	37
17 - 19	14

STEP 1: Find the midpoints

Age	Frequency (f)	Midpoint (x)
8 - 10	12	9
11 - 13	25	12
14 - 16	37	15
17 - 19	14	18

fx = frequency (f) x midpoint (x)

STEP 2: Work out fx

Age	Frequency (f)	Midpoint (x)	fx
8 - 10	12	9	12 x 9 = 108
11 - 13	25	12	25 x 12 = 300
14 - 16	37	15	37 x 15 = 555
17 - 19	14	18	14 x 18 = 252

STEP 3: Work out the total of fx

fx
12 x 9 = 108
25 x 12 = 300
37 x 15 = 555
14 x 18 = 252
= 1215

STEP 4: Work out the total frequency

Frequency (f)
12
25
37
14
= 88

To find a midpoint (x), add the lower and upper limit of the class and divide by 2. e.g. $\frac{8+10}{2}$ = 9

STEP 5: Divide the total fx by the total frequency (f)

1215 ÷ 88 = 13.8

The mean is 13.8 years

Finding the class in which the median lies

We can estimate which grouped class the **median** can be found. We need to work out what position in the data the median will be. If there are n pieces of data, the median value will be in position $\frac{n+1}{2}$.

In this case the median position is $\frac{88+1}{2}$ = 44.5th

Age	Frequency
8 - 10	12
11 - 13	25
14 - 16	37
17 - 19	14

The first row contains the first 12 numbers (or positions).

Up to row 2, 37 (12 + 25) numbers are represented in total.

Therefore the 44.5th number would be found in the third row.

Finding the modal class

The **modal class (group)** is the class with the highest frequency. We cannot say what the single most frequent value was, but we can say which group had the most data in it.

In the table to the left the modal age group is 14-16.

Mathematics

Higher

Unit 23

Grouped frequency diagrams and polygons

Age (in years)	Frequency
0 < a ≤ 10	5
10 < a ≤ 20	14
20 < a ≤ 30	12
30 < a ≤ 40	10
40 < a ≤ 50	7
50 < a ≤ 60	4
60 < a ≤ 70	8

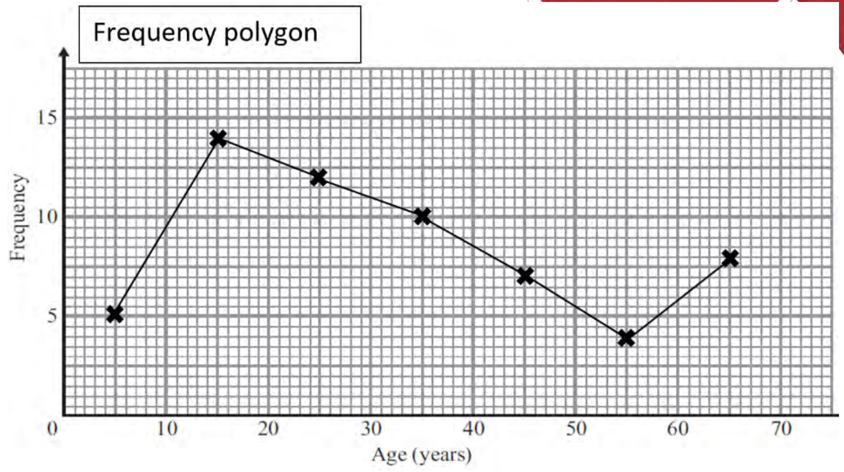
The table shows the age in years of 60 people.
You may be asked to produce a frequency polygon or frequency diagram from this data.
There are distinct differences between the graphs. However, the grouped data is on the x-axis and frequency always goes up the y-axis.

1. Find midpoint of class widths

Age (in years) MP		Frequency
0 < a ≤ 10	5	5
10 < a ≤ 20	15	14
20 < a ≤ 30	25	12
30 < a ≤ 40	35	10
40 < a ≤ 50	45	7
50 < a ≤ 60	55	4
60 < a ≤ 70	65	8

2. Plot relevant frequency at each midpoint

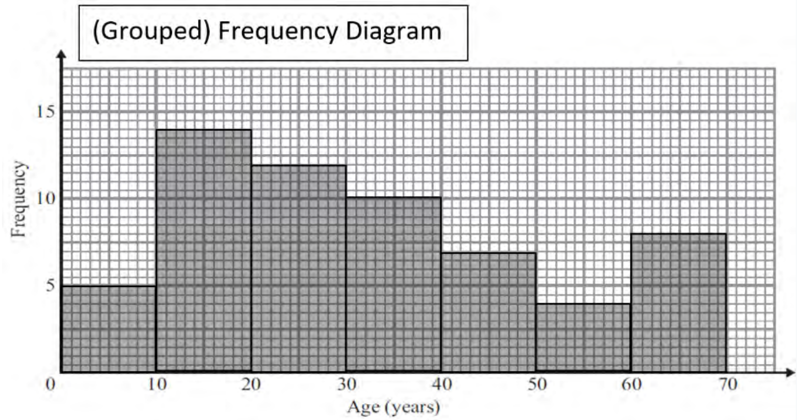
3. Join with straight lines. Start at the first plot and finish on the last plot.



The lowest and highest numbers in the class interval go each side of the bar. The first-class interval is 0 < a ≤ 10, so the first bar starts at 0 and ends at 10.

The bars are touching as it is continuous data.

You will be expected to work backwards from a frequency polygon or diagram to create a grouped frequency table like the one shown on the left. This means you can estimate the mean, median and modal class (see previous page). You can use these measurements to compare different distributions.



Mathematics

Cumulative Frequency

Higher

Unit 24

Cumulative frequency is where you add up frequencies to provide the running total.
You can then use this data to plot a cumulative frequency graph (only used with continuous data).

Method to draw a Cumulative Frequency Graph/Diagram

- To draw a cumulative frequency graph, you need a cumulative frequency table. Sometimes this is given but sometimes it needs to be created from a grouped frequency table.
- Use the upper limit (& less than symbol) of each group from the grouped frequency table in the cumulative frequency table, each frequency is added onto the frequencies before it to create the cumulative frequency.
- Once the cumulative frequency table is completed, the axes & scales can be set up.
- Decide on the scale you are going to use for the cumulative frequency.
- The cumulative frequency is always on the vertical axis and must start from zero. The values are placed on the lines not in the spaces.
- Decide on the scale you are going to use for whatever your graph is about eg. Length of leaves, and place on the horizontal axis.
- The horizontal axis does not need to start from zero. The values are placed on the lines not in the spaces.
- Complete both axes and label fully.
- To draw a cumulative frequency graph, you must read across to the **upper limit** of each group (from the grouped frequency table) and up to its corresponding cumulative frequency. Place a dot at this point.
- Continue to do this until all points have been plotted. Join the dots up in order (1st point to 2nd, 2nd point to 3rd etc) with straight lines. Ensure you use a ruler.
- Do not join the 1st point to the cumulative frequency axis.
- The cumulative frequency graph is used to find an estimate for the median value, the interquartile range and other questions based on the graph.

Example

(a) Draw a cumulative frequency graph for the following.

Weight (w) kg	40 ≤ w < 45	45 ≤ w < 50	50 ≤ w < 55	55 ≤ w < 60	60 ≤ w < 65	65 ≤ w < 70
Frequency	5	11	17	21	24	2

(a) 1st set up a cumulative frequency table

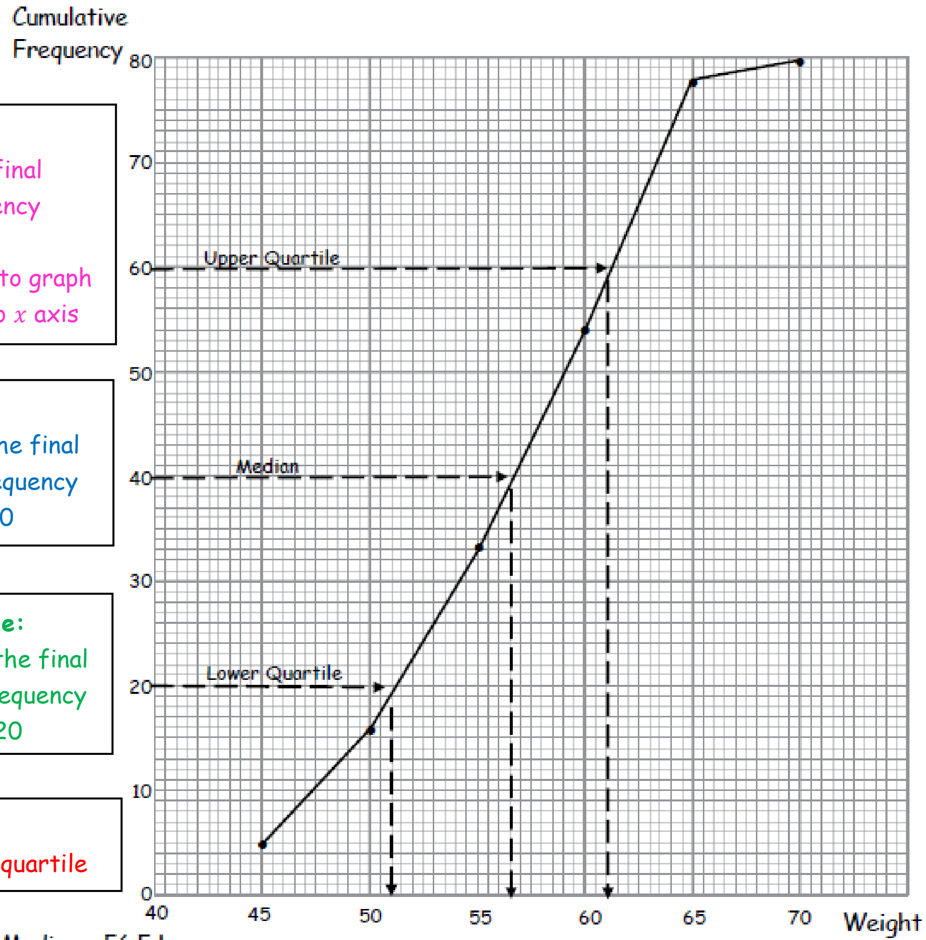
Weight (w) kg	w < 45	w < 50	w < 55	w < 60	w < 65	w < 70
Cumulative Frequency	5	16	33	54	78	80

Upper quartile:
Find 75% of the final cumulative frequency
75% of 80 = 60
Draw dotted line to graph then read down to x axis

Median:
Find 50% of the final cumulative frequency
50% of 80 = 40

Lower quartile:
Find 25% of the final cumulative frequency
25% of 80 = 20

Interquartile range:
Upper quartile – lower quartile



(b) What is the median? Median = 56.5 kg

(c) What is the interquartile range? Interquartile range = 61 – 51 = 10

Mathematics

Higher

Unit 24

Common Types of Question

As well as finding the Median, Lower Quartile, Upper Quartile and Interquartile range, there are other common questions as seen below.

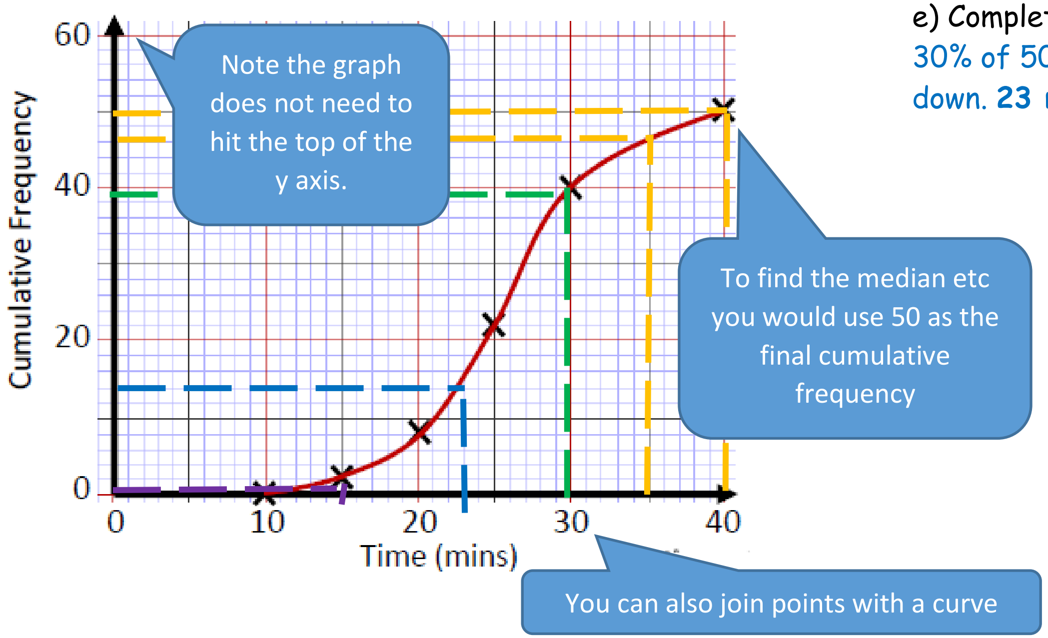
The table shows 50 peoples times in a fun run..

a) Construct a cumulative frequency table

Time (mins)	Frequency
$10 < t \leq 15$	2
$15 < t \leq 20$	6
$20 < t \leq 25$	14
$25 < t \leq 30$	18
$35 < t \leq 40$	10

Time (mins)	CF
$10 < t \leq 15$	2
$10 < t \leq 20$	8
$10 < t \leq 25$	22
$10 < t \leq 30$	40
$10 < t \leq 40$	50

b) Draw the cumulative frequency graph on the grid below.



c) How many runners took more than 30 minutes?

Draw a dotted line up from 30 minutes and read across. Up to 30 minutes, there are 40 runners. Therefore, as there is 50 runners in total, **10 runners took more than 30 minutes**.

d) How many runners took 35 and 40 minutes?

Find the amount of runners at 35 and 40 minutes. 35 minutes = 46 runners. 40 minutes = 50 runners. **Therefore 4 runners took between 35 and 40 minutes**.

e) Complete the sentence; '30% of the runners finished within……minutes'

30% of 50 runners is 15 runners. Draw a dotted line from 15 runners and read down. **23 minutes**.

f) What is the modal group?

You can see from the table that this is **$25 < t \leq 30$ minutes.**

However, it can be observed on the graph as the steepest section.

g) What percentage of runners finished within 15 minutes?

Find how many people finished within 15 minutes, drawing the line shows 2 people. Calculate $\frac{2}{50}$ as a percentage. $\frac{2}{50} = \frac{4}{100} = 4\%$

Mathematics

Higher

Unit 24

Disadvantages of CF graphs

Sometimes you are asked about specific values, however, we do not have the individual data items. We know how many items (e.g. people) were in each category but they might all be at the top end or the bottom end. Without the original list of data, we do not know.

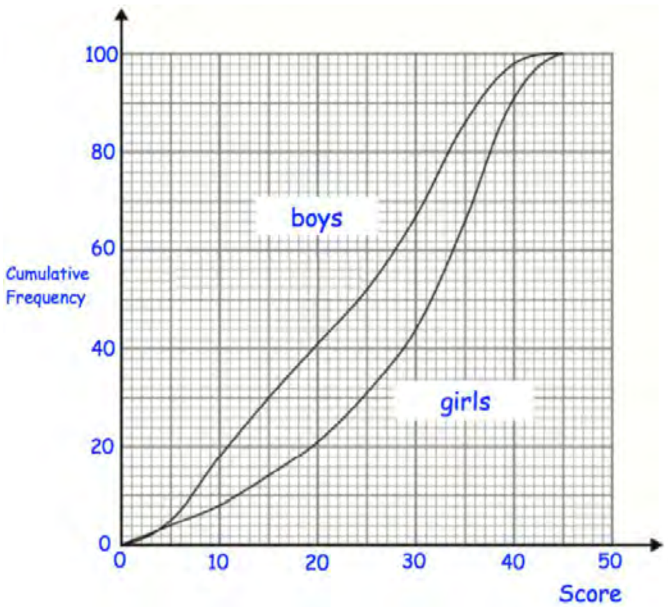
Advantages of CF graphs

There might be too much data therefore using class intervals makes the data far more manageable. It is also easy to find the median etc from the graph and compare distributions.

Interpreting and Comparing Cumulative Frequency Graphs

When you are asked to compare cumulative frequency graphs it is good to talk about the median value (the average) and the interquartile range. A bigger interquartile range means the data is more spread out and less consistent. A smaller interquartile range means the data is close together and therefore more consistent. The smaller the interquartile range, the steeper the mid-section of the graph.

Example: Make two comparisons between the boys and girls scores



Boys' median = 24 IQR = 19   Girls' median = 31.5 IQR = 14.5

The girls scored higher on average with a median of 31.5 compared to the boys' median of 24.

The girls' results were more consistent (less spread out) with a lower interquartile range of 14.5 compared to the boys' interquartile range of 19. This can also be seen as the girls' graph is steeper in the midsection.

Mathematics

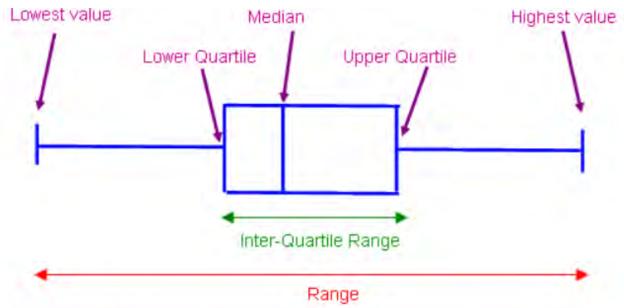
Higher

Unit 25

Box-and-Whisker Plots

Box-and-whisker plots can be used to display discrete and continuous data such as that from a cumulative frequency graph.
They can be drawn either horizontally or vertically.
Start by drawing the median, then lower and upper quartiles to create a 'box'.
Next, draw the lowest and highest values, the 'whiskers'.

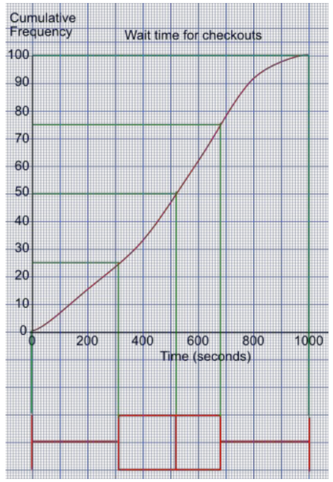
Box-and-whisker plot



Note: The minimum value is the lowest possible value of your first group, and the maximum value is the highest possible value of your last group

Finding the quartiles of continuous data

Read Unit 31 for a refresher on finding the median, lower and upper quartile from a cumulative frequency graph.
You can use these values to draw your box plot.
If you have the chance, draw your box plot directly beneath your cumulative frequency graph, using the same scale on the x axis then you can extend the vertical line downwards saving time!



Finding the quartiles of discrete data. You can also find the median and lower and upper quartile of a set of individual numbers (discrete data).

Position

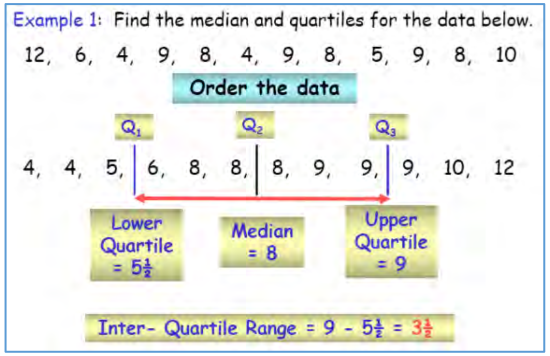
(n= number of items)

Median = $\frac{n+1}{2}$

LQ = $\frac{n+1}{4}$

UQ = $\frac{3(n+1)}{4}$

Remember to place numbers in ascending order!

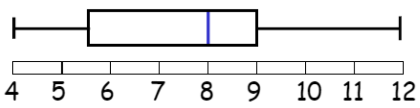


$$Median = \frac{n+1}{2} = \frac{12+1}{2} = 6.5th$$

$$LQ = \frac{n+1}{4} = \frac{12+1}{4} = 3.25th$$

$$UQ = \frac{3(n+1)}{4} = \frac{3(12+1)}{4} = 9.75th$$

Mark these positions on the list of numbers and work out the corresponding numbers (see on diagram).



Once you have found the quartiles of discrete data you can plot a box-and-whisker plot.

Mathematics

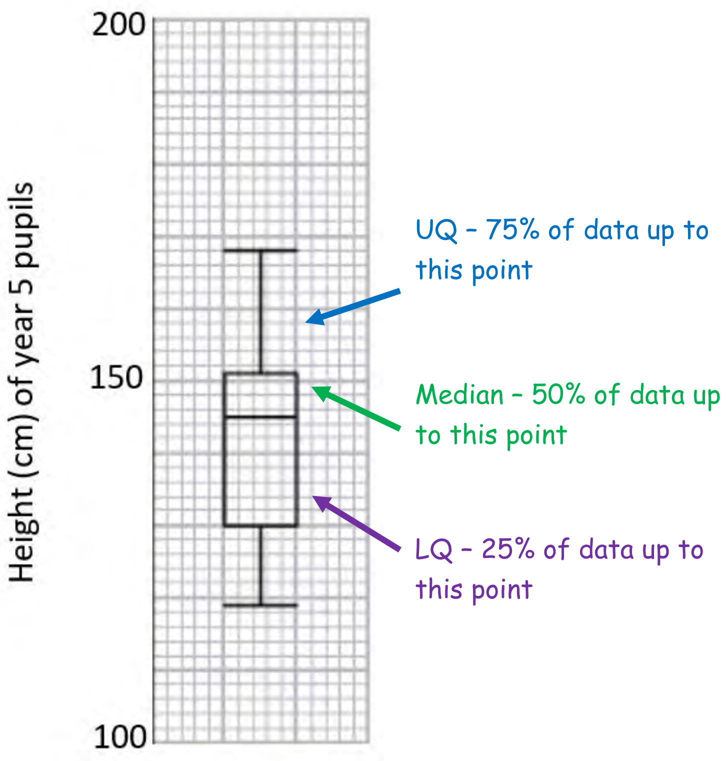
Higher

Unit 25

Terminology check

Q1	Q2	Q3
QUARTILE 1	QUARTILE 2	QUARTILE 3
LOWER QUARTILE	MEDIAN	UPPER QUARTILE
Represents 25% of data	Represents 50% of data	Represents 75% of data

Inter-quartile range = Upper quartile – lower quartile

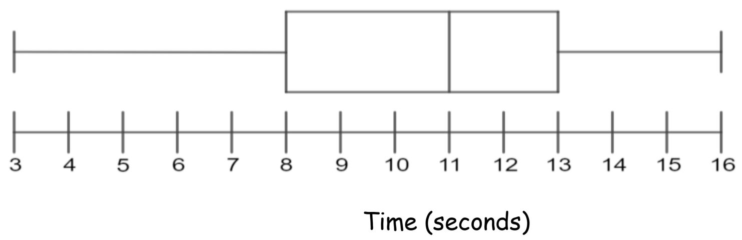


Common Types of Question

1. Which measurement of a boxplot should you use when discussing average? Median
2. If a box-and-whisker diagram represents 300 people, how many people are within the interquartile range?

   50% of people are within the interquartile range – the 'box'

   (Upper Quartile 75% – Lower Quartile 25%) Therefore half of 300 is 150
3. Ann records the time taken to write 200 text messages. The below boxplot represents this data.



a. How many messages took Ann more than 13 seconds to write?

13 seconds represents the UQ (75%) so 25% of 200 took more than 13 second which is 50 messages

b. What fraction of texts took less than 8 seconds to write?

Up to 8 seconds represents 25% of data (LQ) equivalent to $\frac{1}{4}$

Mathematics

Higher

Unit 25

Comparing Box-and-Whisker Plots

Box-and-whisker plots are especially useful when you want to compare two distributions.

Note: you cannot tell the sample size by looking at a boxplot; it is based on percentages of the sample size, not the sample size itself!

Central Tendency

A measure of **central tendency** is a single value that attempts to describe a set of data by identifying the central position within that set of data. The mean, median and mode are all valid measures of central tendency, but under different conditions, some measures of central tendency become more appropriate to use than others (see Unit 30 for more information on this).

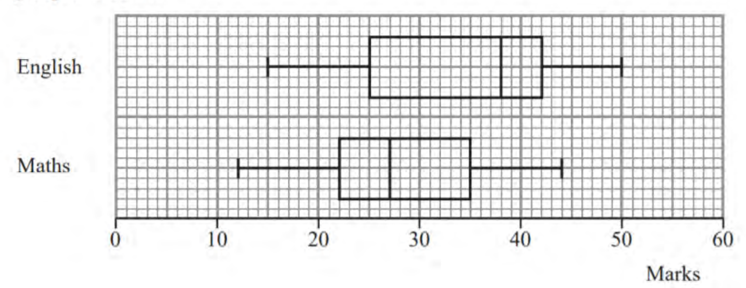
The median is the measure of **central tendency** on a Box-and-Whisker diagram. It tells us something valuable about the data – roughly what values we can expect in the middle. Importantly, unlike the mean, it is not affected by outliers or extreme values (data points which sit far away from all the others).

Measure of Spread

On a box-and-whisker plot, the range and interquartile range can help us analyse the spread of data. The higher the range and interquartile range the higher variation in data. The smaller they are, the more consistent the data is. One limitation of the range is that it is affected by outliers. Fortunately, the interquartile range is much better as it is unaffected by any outliers and this is why the IQR is the preferred measure of spread.

Example: When you are asked to compare distributions, you need to state which one has a greater spread (by looking at the IQR and/or the range) and which one has a higher average (by looking at the median).

The box plots show the distribution of marks in an English test and in a Maths test for a group of students.



The Median mark for English was 38 and Maths was 27. Therefore, on average, students scored better in the English test.

The interquartile range for English was 17 and Maths was 13. Therefore, the Maths marks are more consistent.

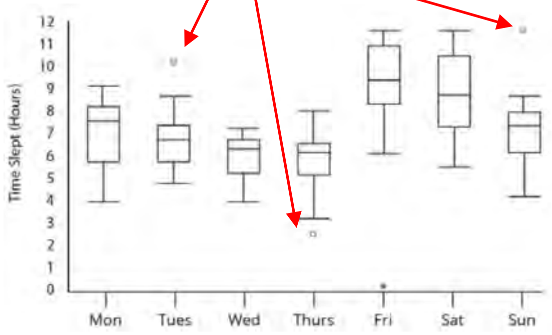
Mathematics

Higher

Unit 25

Outliers

Outliers are extreme low or high values. When reviewing a box-and-whisker plot, an outlier is defined as a data point that is located outside the whiskers of the box plot. As seen on the diagram below, they are shown as dots outside the main box plot.



You can decide if a data point is an outlier by following this method:

1. Calculate the interquartile range of your data.
2. Multiply the interquartile range by 1.5.
3. Subtract this value away from the lower quartile. Any data points below this are outliers.
4. Add this value to the upper quartile. Any data points below this are outliers.

Example: How many of these data points would be outliers?

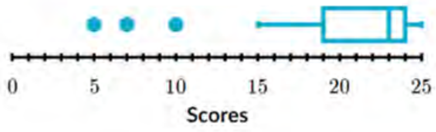
The test scores of a class of year 10 pupils

5 7 10 15 (19) 21 21 22 22 (23) 23 23 23 23 (24) 24 24 24 25

LQ 19 — Median 23 — UQ 24

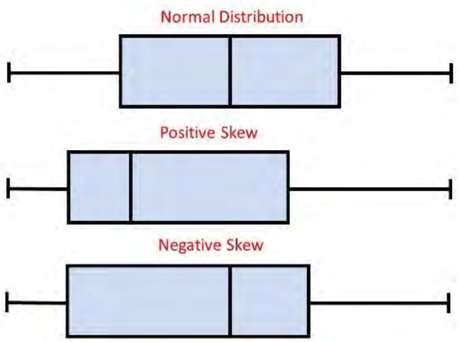
1. IQR = 24-19 = 5
2. 5 x 1.5 = 7.5
3. 19 – 7.5= 11.5
4. 24 + 7.5 = 31.5

5, 7, and 10 would be plotted as outliers



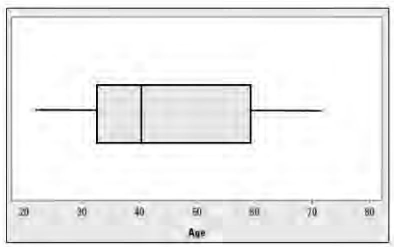
Skewed Data

Skewed data show a lopsided boxplot, where the median cuts the box into two unequal pieces. If the longer part of the box is to the right (or above) of the median, the data is said to be skewed right (positive). If the longer part is to the left (or below) of the median, the data is skewed left (negative).



If one side of the box is longer than the other, it does not mean that side contains more data. Instead, it indicates a wider range in the values of data in that section (meaning the data are more spread out). A smaller section of the boxplot indicates the data are more condensed (closer together).

In the figure to the right, the boxplot represents employee ages in a company. The ages are skewed right (positive). The part of the box to the left of the median (representing the younger employees) is shorter than the part to the right of the median (representing the older employees). That means the ages of the younger employees are closer together than the ages of the older employees.



Mathematics

Higher

Unit 26

Constructing and Interpreting Graphs in Everyday Life

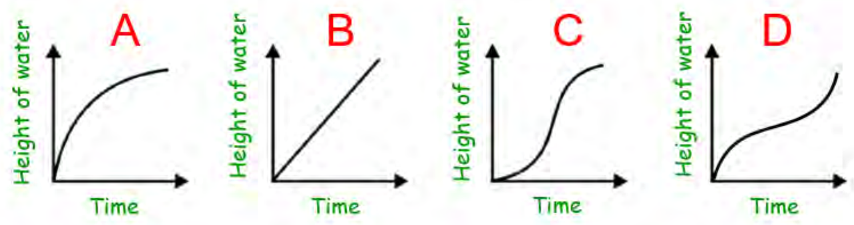
Often you will be presented with a "real life" graph and asked a few questions based upon it.

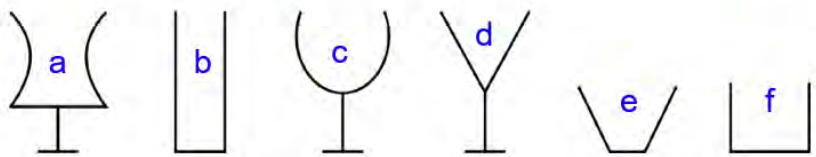
Method for Interpreting Real-Life Graphs

- Look carefully at both **axes** to see what the variables are
- Look at the **scale** carefully so you can accurately read the graph
- Look at the **gradient** of the graph:
  What does a horizontal line mean?
  What does a positive/negative slope mean?
- Always **read** the question carefully and **check** your answers.

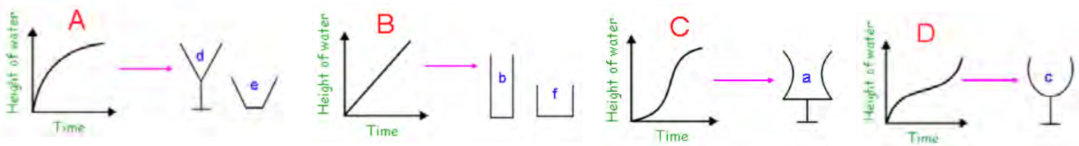
Example – Story Graph

Water is poured into various glasses at a constant rate. The graphs below are sketches showing how the height of water in the glasses' changes over time. Match up the shape of the glasses with their graphs
Note: Each graph can represent more than one glass.





Answers:



- Look carefully at both **axes** to see what the variables are
  We have height of water going up the y-axis, and time going along the x-axis
- Look at the **scale** carefully so you can accurately read the graph
  There is no scale, so this doesn't apply
  Note: This is also the reason why more than one glass can match to each graph
- Look at the **gradient** of the graph:
  What does a **straight**-line mean?
  The height of the water is changing by the same amount as time passes, so the sides of the glass must be straight!

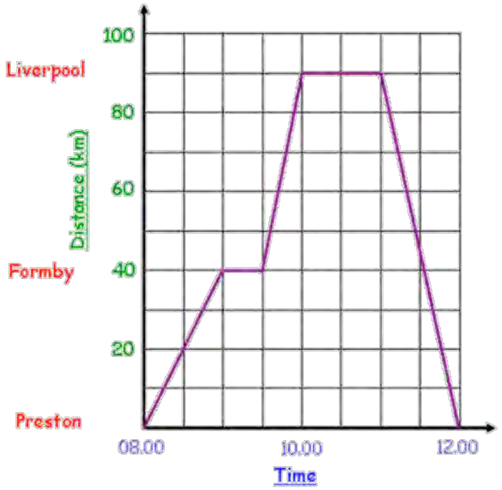
  What does a **curved** line mean?
  Well, it depends on the shape of the curve, but generally a curved line means that the height of the water is not changing by the same amount, so the sides of the glass must also be curved
- Try to picture that water dropping constantly into those glasses and what the height of the water will be doing.

Mathematics

Higher

Unit 26

Example – Travel Graph



The graph on the left shows a journey made by a family in a car between Preston, Formby and Liverpool. Look at the graph and then answer the following questions:

(a) What time did the family arrive in Liverpool?

(b) What is the distance from Formby to Liverpool?

(c) How long did the family spend not moving?

(d) What was the average speed on the journey home?

- Look carefully at both **axes** to see what the variables are
  We have distance in kilometres going up the y-axis, and time in hours going along the x-axis
- Look at the **scale** carefully so you can accurately read the graph
  On the y-axis every square represents 10km, and on the x-axis every square is 15 minutes
- Look at the **gradient** of the graph
  What does a **horizontal** line mean?
  A horizontal line means that time is still passing, but the distance travelled is not changing, so the family must have stopped moving.

  What does a **positive/negative slope** mean?
  A positive slope means the family are travelling from Preston towards Liverpool, and a negative slope means they are on their way back home.

  Note: You could say that the family are travelling faster between Formby and Liverpool than between Preston and Formby, we know this because the line is steeper meaning they are travelling more distance in less time, so they must be going faster.
- We can now answer all the questions.

Answers:

(a) What time did the family arrive in Liverpool?

The line first hits Liverpool at 10.00

(b) What is the distance from Formby to Liverpool?

Formby is 40km from Preston, Liverpool is 90km from Preston, so the distance from Formby to Liverpool must be 50km.

(c) How long did the family spend not moving?

When the family is not moving we see a horizontal line. That happens twice, firstly at Formby for 30 minutes, and then at Liverpool for 60 minutes, giving us a total of 90 minutes, or one and a half hours.

(d) What was the average speed on the journey home?

Using the formula: Speed = Distance ÷ Time

On the journey home we have: Speed = 90km ÷ 1 hour

= 90 km/hr

Mathematics

Higher

Unit 26

Example – Conversion Graph

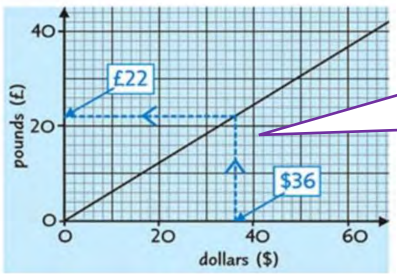
A conversion graph is used to change one unit into another.

This could be changing between miles and kilometres, pounds to a foreign currency, or the cost of a journey based on the number of miles travelled.

Method for using conversion graphs:

1. Draw a line from a value on one axis – keep going until you hit the line.
2. Change direction and go straight to the other axis – the value you get on this axis is equivalent (the same as) to the value on the other

Example: Doug went on holiday to South Carolina and paid **$360** for a PlayStation. On the way back Doug saw the same PlayStation in Cardiff Airport for **£250**. Did Doug get a good deal while on holiday?



Make sure you draw your conversion lines on the graph – these are your **workings**.

Answering the question:

$360 dollars isn't on the graph, so you need to find a way of making the calculation as easy as possible for yourself. In this question the easiest way is to read off the value for $36 and then multiply by 10 (because $36 x 10 = £360).

Reading off the graph: $36 = £22

So $360 would be: £22 x 10 = £220

To finish you need to compare the values and add a conclusion:

£220 is less than £250, so Doug got the best deal as the PlayStation was cheapest in South Carolina.

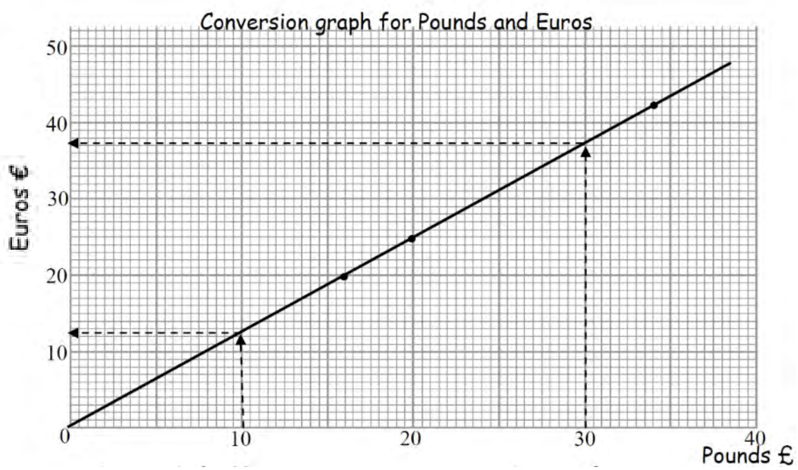
Method to draw a Conversion Graph

- For a conversion graph you need at least 3 pairs of values that are equivalent to each other. Eg one pair could be 1 inch ≡ 2.54 cm
- Decide on the scale you are going to use for the 1st set of data. This is usually on the horizontal axis.
- Decide on the scale you are going to use for the 2nd set of data. This is usually on the vertical axis.
- The vertical axis does not have to have the same scale as the horizontal axis but each axis must have a "uniform scale".
- Each axis should start from zero.
- The values are placed on the lines not in the spaces.
- Complete both axes and label fully.
- Plot each point by reading across to its horizontal value and up to its corresponding vertical value. Mark the position with either a cross or a dot.
- Once all the points have been plotted join them up with a straight line that **passes through all** the points.
- The conversion graph can then be used to answer questions such as converting from one value to another.
- Write a title for your conversion graph.

Example

(a) Draw a conversion graph for

Pounds £	16	20	34
Euros €	20	25	42.50



b) Using the graph, find how many Euros are equivalent to £30.

€ 37.50

c) How many Euros are equivalent to £50? Explain fully how you got your answer.

Find how many Euros are equivalent to £10 and then multiply by 5

£10 = €12.50

£50 = 12.50 x 5 = € 62.50

Mathematics

Higher

Unit 27

Perimeter, Area, Volume and Density

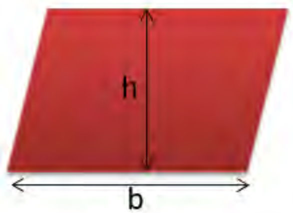
The **area** of a 2D shape is the space inside it. It is measured in units squared e.g. cm^2

The **perimeter** of a shape is the distance around the edge of the shape. Units of length are used to measure perimeter e.g. mm, cm, m

A **compound shape** is a shape made from other shapes joined together.

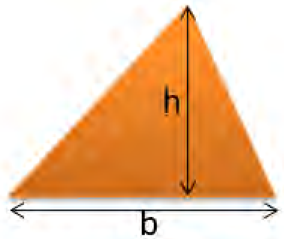
Formulas for Area:

Parallelogram



A = b × h

Triangle



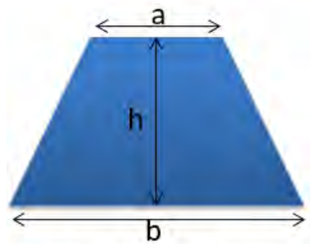
$A = \frac{b \times h}{2}$

Note: You must use the perpendicular height

Rectangle / Square

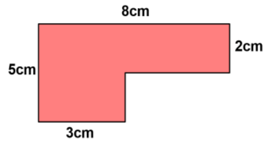
A = l × w

Trapezium



$A = \frac{(a+b) \times h}{2}$

Example 1: Find the perimeter and area of the compound shape



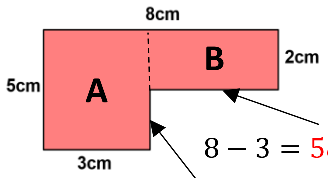
Area

Step 1: Split the shape into shapes that you can find the area of

Step 2: Find the missing lengths of sides

Step 3: Work out the area of each shape

Step 4: Work out the total area, remembering the units



$8 - 3 = 5cm$

$5 - 2 = 3cm$

$\text{Area A} = (5 \times 3) = 15cm^2$

$\text{Area B} = (2 \times 5) = 10cm^2$

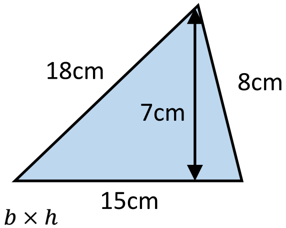
$\text{Total area} = 15 + 10 = 25cm^2$

Perimeter

Add up all the outside edges

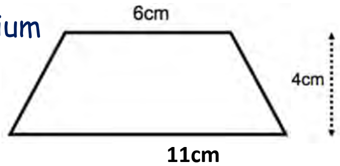
$\text{Perimeter} = 3 + 5 + 8 + 2 + 5 + 3 = 26cm$

Example 2: Find the area of the triangle



$\text{Area} = \frac{b \times h}{2} = \frac{15 \times 7}{2} = 52.5cm^2$

Example 3: Find the area of the trapezium



$\text{Area} = \frac{(a+b) \times h}{2} = \frac{(6+5) \times 4}{2} = 22cm^2$

Mathematics

Higher

Unit 27

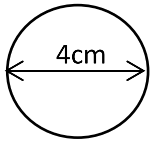
The circumference of a circle is the distance around the outside of the circle and is calculated using the formula:

$$Circumference = \pi \times d$$

The area of a circle is calculated using the formula:

$$Area = \pi \times r^2$$

Example 1: Find the circumference of a circle with a diameter of 4cm.



C = $\pi \times 4$

= 4π

or = 12.57cm

Example 2: Find the diameter of a circle with a circumference of 20cm.

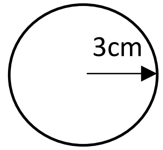
C = $\pi \times d$

20 = $\pi \times d$

$\frac{20}{\pi} = d$

$6.37cm = d$

Example 3: Find the area of a circle with radius 3cm



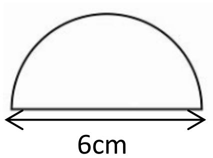
A = $\pi \times r^2$

= $\pi \times 3^2$

= 9π

or = 28.3cm^2

Example 4: Find the perimeter of the semicircle.



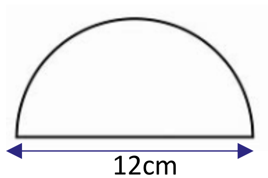
$$P = \frac{\pi \times d}{2} + d$$

$$P = \frac{\pi \times 6}{2} + 6$$

$P = 3\pi + 6$

P = 15.42cm

Example 5: Find the area of the semicircle.



$$A = \frac{\pi \times r^2}{2}$$

$$A = \frac{\pi \times 6^2}{2}$$

$A = 56.55cm^2$

Example 6: Find the radius of a circle when the area is 20cm^2.

A = $\pi \times r^2$

20 = $\pi \times r^2$

$\frac{20}{\pi} = r^2$

$\sqrt{\frac{20}{\pi}} = r$

$2.52cm = r$

Mathematics

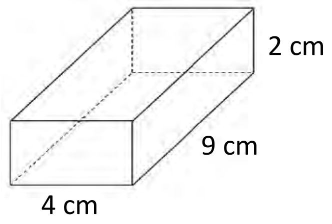
Higher

Unit 27

Surface Area

The **surface area** of an object is the area of each face added together.
It is measured in units squared e.g. cm^2.

Example 1: Find the surface area of the cuboid



Surface area = total area of each face

Area of each face:

$Front = 4 \times 2 = 8$

$Back = 4 \times 2 = 8$

$Left\ Side = 9 \times 2 = 18$

$Right\ Side = 9 \times 2 = 18$

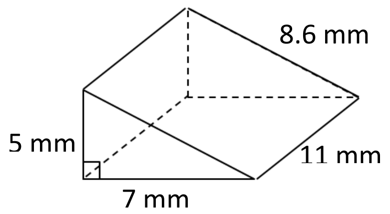
$Bottom = 4 \times 9 = 36$

$Top = 4 \times 9 = 36$

$Total = 8 + 8 + 18 + 18 + 36 + 36$

$= \mathbf{126cm^2}$

Example 2: Find the surface area of the triangular prism



Surface area = total area of each face

Area of each face:

$Front = \frac{7 \times 5}{2} = 17.5$

$Back = \frac{7 \times 5}{2} = 17.5$

$Side = 5 \times 11 = 55$

$Bottom = 7 \times 11 = 77$

$Top = 11 \times 8.6 = 94.6$

$Total = 17.5 + 17.5 + 55 + 77 + 94.6$

$= \mathbf{261.6mm^2}$

Mathematics

Higher

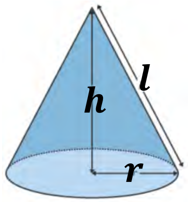
Unit 27

Formulas for Surface Areas of More Complex Shapes

Cone

The surface area of a cone is the area of the circular base, πr^2, plus the area of the curved surface, πrl. It is calculated using the formula:

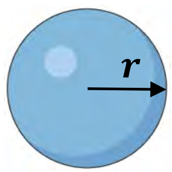
$surface\ area\ of\ a\ cone = \pi r^2 + \pi rl$



Sphere

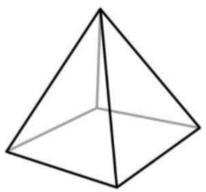
The surface area of a sphere is calculated using the formula:

$surface\ area\ of\ a\ sphere = 4\pi r^2$



Pyramid

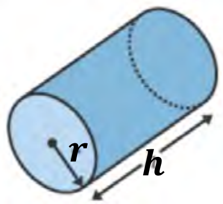
The surface area of a pyramid can be calculated by adding the area of the base to the sum of the areas of the triangular faces.

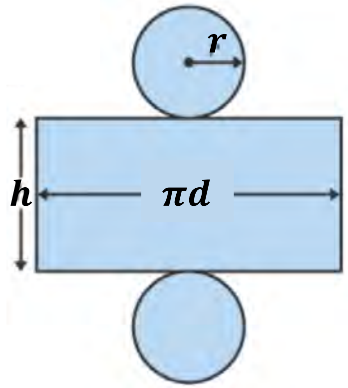


Cylinder

The surface area of a cylinder is the area of the 2 circular faces at the ends, $2 \times \pi r^2$, plus the area of the curved surface, πdh.

The net of a cylinder can help visualise the surface area. The curved surface has been opened out to form a rectangle, where the width of the rectangle is the height of the cylinder, and the length of the rectangle is the same as the circumference of the circle, πd.





The surface area of a cylinder is calculated using the formula:

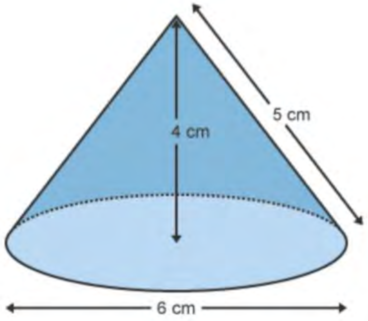
$surface\ area\ of\ a\ cylinder = 2\pi r^2 + \pi dh$

Mathematics

Higher

Unit 27

Example 1: Calculate the surface area of the cone.
Give your answer to 1 decimal place.



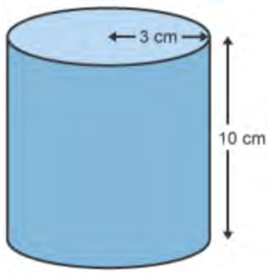
From the diagram, $r = 3cm$ (half of the diameter), and $l = 5cm$

$surface\ area\ of\ a\ cone = \pi r^2 + \pi rl$

$= \pi \times 3^2 + \pi \times 3 \times 5$

$= 75.4cm^2$ (to 1 d.p.)

Example 2: Calculate the surface area of the cylinder.
Give your answer to 3 significant figures.



From the diagram, $r = 3cm$, $d = 6cm$, and $h = 10cm$

$Area\ of\ each\ circular\ end = \pi r^2$

$= \pi \times 3^2$

$Area\ of\ the\ curved\ surface\ area = \pi dh$

$= \pi \times 6 \times 10$

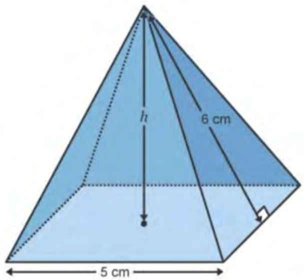
$Total\ surface\ area = \pi \times 3^2 + \pi \times 3^2 + \pi \times 6 \times 10$

$or\quad 2(\pi \times 3^2) + \pi \times 6 \times 10$

$= 78\pi cm^2$

$= 245cm^2$ (to 3 sf)

Example 3: Calculate the surface area of the square-based pyramid.



$Area\ of\ the\ base = 5 \times 5$

$= 25cm^2$

$Area\ of\ each\ triangular\ face = \frac{b \times h}{2}$

$= \frac{1}{2} \times 5 \times 6$

$= 15cm^2$

There are four identical triangular faces.

$Total\ surface\ area = 25 + 4 \times 15$

$= 85cm^2$

Example 4: Calculate the surface area of a football with a radius of 12cm.
Give your answer to 1 decimal place.

From the question, $r = 12cm$

$surface\ area\ of\ a\ sphere = 4\pi r^2$

$= 4 \times \pi \times 12^2$

$= 1{,}809.6cm^2$ (to 1 d.p.)

Mathematics

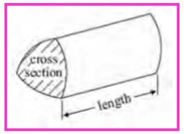
Higher

Unit 27

Volume

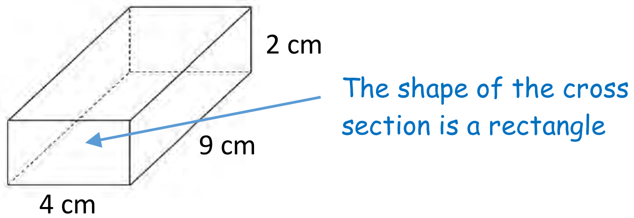
- The **volume** of an object is the amount of space that it occupies. It is measured in units cubed e.g. cm^3.
- To calculate the volume of any prism we use the formula:

$$volume = area\ of\ cross\ section\ \times length$$



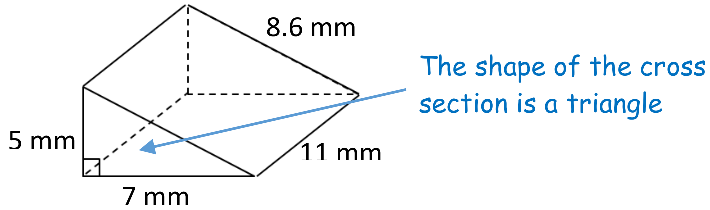
- A **prism** is a 3D shape which has a continuous cross-section.

Example 1: Find the volume of the cuboid



$$\boldsymbol{Volume} = area\ of\ c.s. \times length$$
$$= 4 \times 9 \times 2$$
$$= \mathbf{72cm^3}$$

Example 2: Find the volume of the triangular prism



Area of cross-section:

$$Area\ of\ triangle = \frac{5 \times 7}{2}$$
$$= 17.5mm^2$$

$$\boldsymbol{Volume} = area\ of\ c.s. \times length$$
$$= 17.5 \times 11$$
$$= \mathbf{192.5mm^3}$$

Mathematics

Higher

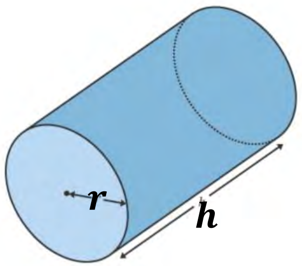
Unit 27

Formulas for Volumes of More Complex Shapes

Cylinder

The volume of a cylinder is calculated using the formula:

$volume\ of\ a\ cylinder = \pi r^2 h$

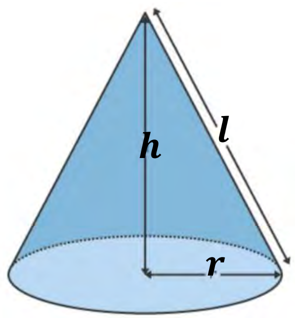


Cone

The volume of a cone is one third the volume of a cylinder

It is calculated using the formula:

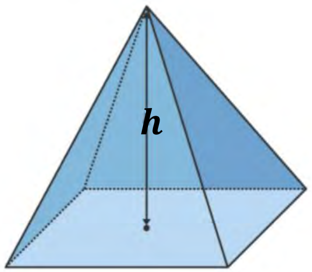
$volume\ of\ a\ cone = \frac{1}{3}\pi r^2 h$



Pyramid

The volume of a pyramid can be calculated using the formula:

$volume\ of\ a\ pyramid = \frac{1}{3} \times area\ of\ base \times perpendicular\ height$

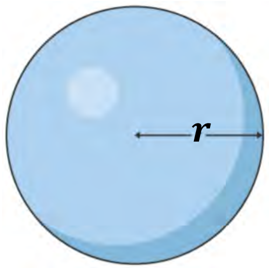


The perpendicular height, h, is the height of the pyramid measured at a right-angle from the base.

Sphere

The volume of a sphere is calculated using the formula:

$volume\ of\ a\ sphere = \frac{4}{3}\pi r^3$

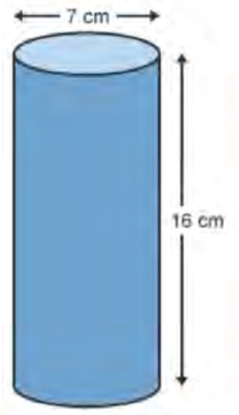


Mathematics

Higher

Unit 27

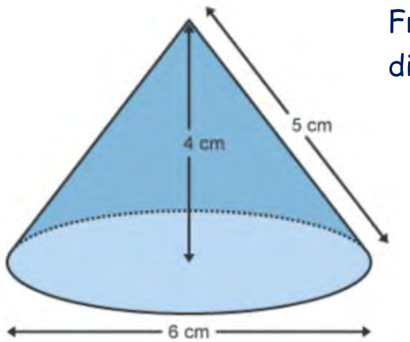
Example 1: Calculate the volume of the cylinder.



From the diagram $r = 3.5cm$ (half the diameter), and $h = 16cm$

$$volume\ of\ a\ cylinder = \pi r^2 h$$
$$= \pi \times (3.5)^2 \times 16$$
$$= 615.8cm^3 \quad \text{(to 1 d.p.)}$$

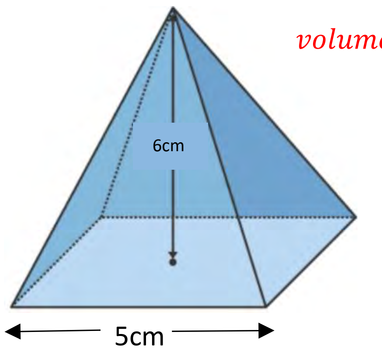
Example 2: Calculate the volume of the cone.



From the diagram $r = 3cm$ (half the diameter), and $h = 4cm$

$$volume\ of\ a\ cone = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \pi \times 3^2 \times 4$$
$$= 37.7cm^3 \quad \text{(to 1 d.p.)}$$

Example 3: Calculate the volume of the square-based pyramid.



$$volume\ of\ a\ pyramid = \frac{1}{3} \times area\ of\ base \times perpendicular\ height$$
$$= \frac{1}{3} \times 5 \times 5 \times 6$$
$$= 50cm^3$$

Example 4: Calculate the volume of a football with a radius of 12cm.
Give your answer to 1 decimal place.

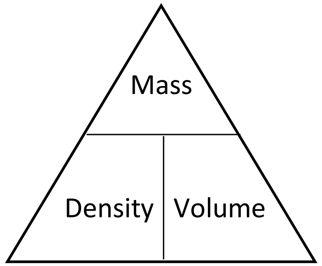
$$volume\ of\ a\ sphere = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 12^2$$
$$= 7{,}238.2cm^3 \quad \text{(to 1 d.p.)}$$

Mathematics

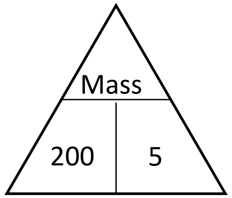
Higher

Unit 27

Density, Mass, and Volume



Example 1: A $5m^3$ box has a density of $200g/m^3$. What is the mass of the box?

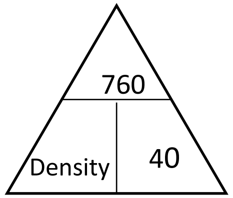


$$Mass = Density \times Volume$$

$$Mass = 200 \times 5 = 1000g$$

Example 2: A piece of gold has a mass of 760 grams and a volume of $40cm^3$.

Work out the density of the piece of gold in kg/m^3.



$$density = \frac{mass}{volume}$$

$760g = 0.76kg$, $40cm^3 = 0.00004m^3$

$$density = \frac{0.76}{0.00004}$$

$$density = 19000\ kg/m^3$$

Example 3:

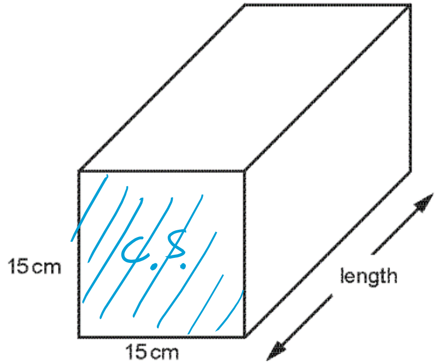


Diagram not drawn to scale

The solid block shown above is made from a metal that has a density of $2.7 g/cm^3$.
The volume of the solid block is $40500 cm^3$.

A hole is drilled through the entire length of the block.
The hole has a cross-sectional area of $25 cm^2$.
Calculate the mass of the block that remains.

$$Volume = area\ of\ c.s. \times length$$

$$40{,}500 = 15 \times 15 \times length$$

$$length = \frac{40{,}500}{15 \times 15}$$

$$length = 180cm$$

$$Volume\ of\ hole = 25 \times 180$$

$$= 4500cm^3$$

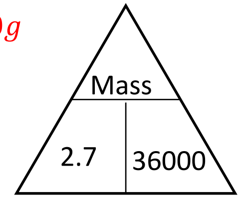
$$Remaining\ volume = 40{,}500 - 4500$$

$$= 36{,}000cm^3$$

$$mass = density \times volume$$

$$= 2.7 \times 36000$$

$$= 97200g$$



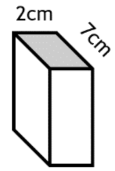
Mathematics

Problem Solving

Higher

Unit 27

Example 1: The volume of this cuboid is 56cm^3.
Find the height of the cuboid



$$Volume\ of\ cuboid = l \times w \times h$$

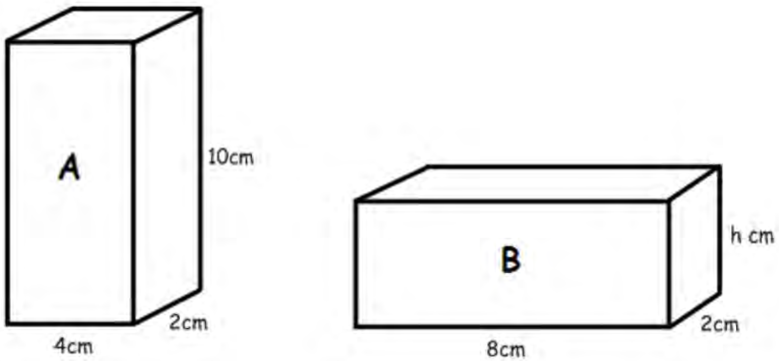
$$56 = 7 \times 2 \times h$$

$$56 = 14 \times h$$

$$\frac{56}{14} = h$$

So, $h = 4cm$

Example 2: The two cuboids A and B, each have the same volume.



Work out the height, $h\ cm$ of cuboid B.

Volume of A: $Volume = 4 \times 2 \times 10$

$$= 80cm^3$$

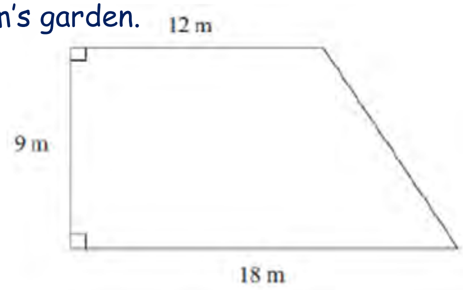
Volume of B: $80 = 8 \times 2 \times h$

$$80 = 16h$$

$$\frac{80}{16} = h$$

So, $h = 5cm$

Example 3: Here is a diagram of Jim's garden.



Jim wants to cover his garden with grass seed to make a lawn.
Grass seed is sold in bags. There is enough grass seed in each bag to cover 20m^2 of garden.
Each bag of grass seed costs £4.99.

Work out the cost of putting grass seed on Jim's garden.

Area of garden (trapezium): Area $= \frac{(a+b) \times h}{2}$

$$= \frac{(12+18) \times 9}{2}$$

$$= 135m^2$$

Grass seed comes in bags that cover 20m^2, so Jim will need 7 whole bags (that will cover 140m^2, he cannot buy less as he can't buy part of a bag).

$$Cost\ of\ grass\ seed = 7 \times 4.99$$

$$= £34.93$$

Mathematics

Higher

Unit 28

Dimensions of Formulae

The advantage of knowing this is that when we are given a formula, we can tell whether it is one for LENGTH, AREA, VOLUME, or neither

We talk about dimensions in terms of objects:

One Dimension (1D)

Objects have just a LENGTH

Units of measurement include:

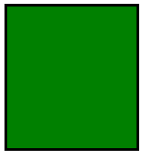
cm, mm, km, m, mile, etc

Two Dimensions (2D)

Objects have an AREA

Units of measurement include:

cm^2, mm^2, km^2, m^2, etc



Three Dimensions (3D)

Objects have a VOLUME

Units of measurement include:

cm^3, mm^3, km^3, m^3, etc

Using Dimensions to Discover what a Formula Represents

1. Change all the variables in the formula to the letter **L**

Note: Variables are just letters that represent lengths, widths and heights

2. Ignore all numbers (apart from powers) and constants

Note: If a letter represents a constant instead of a variable, it will tell you in the question

Remember: pi (π) is just a number!

3. You should now be left with an expression just containing L's, which you can simplify

Important: When you are simplifying, DO NOT cancel anything out.

4. Look at what you are left with. If the formula only contains:

L - this is a formula for length

L^2 - this is a formula for area

L^3 - this is a formula for volume

Any combination - this formula is for none of these.

Mathematics

Higher

Unit 28

Example 1: $7h(l-w)+2w^2$ — $7h(l-w)+2w^2$

1. Change all the variables to the letter L — $7L(L-L)+2L^2$

2. Ignore all numbers (apart from powers) and constants — $L(L-L)+L^2$

3. We are now left with an expression just containing L's, — $L^2-L^2+L^2$

we can multiply out the brackets but do not cancel anything out

4. We are left with: — $L^2-L^2+L^2$

Which means this is a formula for: AREA

Example 2: $\frac{5h^3+2lw^2-hlw}{6}$ — $\frac{5h^3+2lw^2-hlw}{6}$

1. Change variables to L — $\frac{5L^3+2LL^2-LLL}{6}$

2. Ignore all numbers and constants — L^3+LL^2-LLL

3. Simplify — $L^3+L^3-L^3$

4. We are left with: — $L^3+L^3-L^3$

Which means this is a formula for: VOLUME

Example 3: $\frac{2}{3}h(lh+\pi w-h^2)$

1. Change variables to L — $\frac{2}{3}L(LL+\pi L-L^2)$

2. Ignore all numbers and constants — $L(LL+L-L^2)$

3. Simplify — $L^3+L^2-L^3$

4. We are left with: — $L^3+L^2-L^3$

As this is a mix of L^2 and L^3 , this means

this is a formula for: NONE / NEITHER

Example 4: $\frac{kl^3+\pi hw^2}{8hl}$

1. Change variables to L — $\frac{kL^3+\pi LL^2}{8LL}$

(remember k is a constant not a variable)

2. Ignore all numbers and constants — $\frac{L^3+LL^2}{LL}$

3. Simplify the top and bottom separately first: — $\frac{L^3+L^3}{L^2}$

Then simplify by doing the division: — $L+L$

4. We are left with: — $L+L$

Which means this is a formula for: LENGTH

Mathematics

Higher

Unit 29

Compound Measures

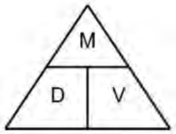
How to use Formula Triangles:

1) **COVER** the thing you want to find and **WRITE DOWN** what's left showing.
2) Now **SUBSTITUTE** in the things you know and **SOLVE**.

Density

You may have come across density in Physics. Density is the mass per unit of volume of a substance. It is typically measured in kg/m³ or g/cm³.

Formula Triangle:



$$Mass = Density \times Volume$$

$$Density = \frac{Mass}{Volume} \qquad Volume = \frac{Mass}{Density}$$

Example: A giant bar of chocolate has a density of 1.4 g/cm³. If the volume of the bar is 1700 cm³, what is the mass of the bar?

Step 1: Use the formula triangle to write down the correct formula:

$$Mass = Density \times Volume$$

Step 2: Substitute the values from the question:

$$mass = 1.4\ g/cm^3 \times 1700\ g$$

Step 3: Solve and check you have used the correct units:

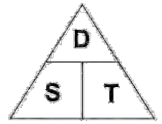
$$mass = 2380g$$

If asked for in kg: $mass\ in\ kg = 2380g \div 1000 = 2.34kg$

Speed

Speed is the distance travelled e.g. the number of km per hour or metres per second.

Formula Triangle:



Distance, Speed, & Time

$$Distance = Speed \times Time$$

$$Speed = \frac{Distance}{Time} \qquad Time = \frac{Distance}{Speed}$$

Example: A car travels 10 miles at 45 miles per hour. How long does this take?

Step 1: Use the formula triangle to write down the correct formula:

$$Time = \frac{Distance}{Speed}$$

Step 2: Substitute the values from the question:

$$Time = \frac{9\ miles}{36\ mph}$$

Step 3: Solve and check you have used the correct units:

$$Time = 0.25\ hours$$

Population density tells us roughly how many people live in an area. We use it to compare how 'built up' two areas are and is commonly measured in population per square kilometre.

Equation:

$$population\ density = \frac{population}{area}$$

Example: The world's population is approximately 7 500 000 000 and the Earth's land area is 150 000 000km².

So, the world's population density = $\frac{7500000000}{150000000}$ = 50 per km²

Fuel Efficiency is usually measured in miles per gallon.

We use the following:

$$distance\ travelled = fuel\ efficiency \times fuel\ used$$

$$fuel\ efficiency = \frac{distance\ travelled}{fuel\ used} \qquad fuel\ used = \frac{distance\ travelled}{fuel\ efficiency}$$

Mathematics

Higher

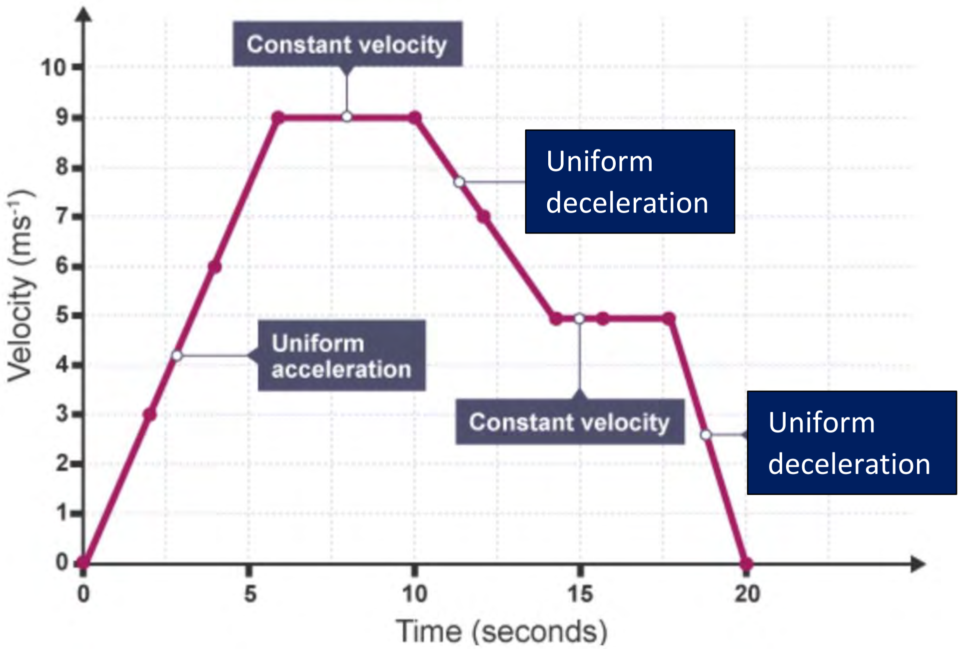
Unit 30

Velocity-Time Graphs

The velocity of an object is its speed in a particular direction.

Velocity is defined as the rate of travel of an object along its direction.

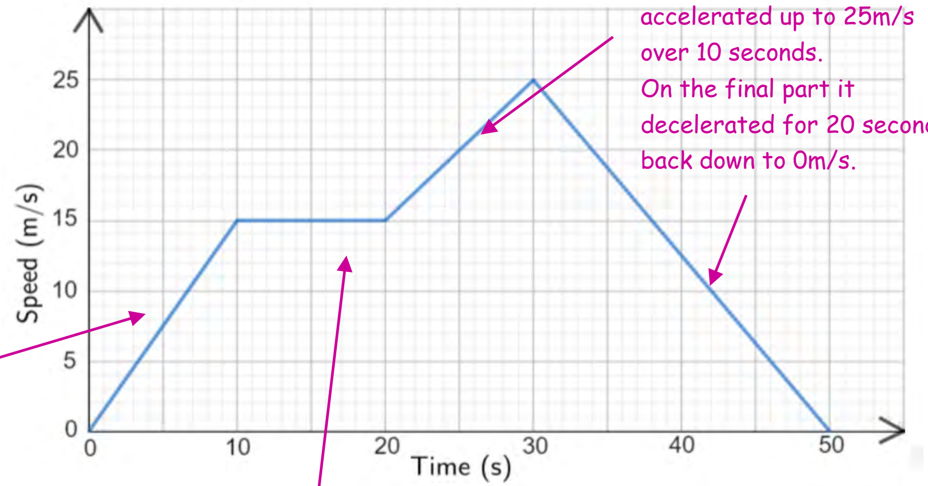
Velocity tells you how fast an object is moving as well as in what direction it is moving.



Deceleration will be represented by a negative as the speed is slowing down.

Example: The speed-time graph below represents a 50-second car journey. Work out the maximum acceleration of the car.

On the first part of the journey, the car accelerated from 0 to 15m/s over the first 10 seconds, the straight line shows that the acceleration is constant.



On the third part, the car accelerated up to 25m/s over 10 seconds.

On the final part it decelerated for 20 seconds back down to 0m/s.

On the second part of the journey, the line is flat, meaning the car's speed did not change for 10 seconds – it was moving at **constant speed**.

Acceleration is the change in velocity divided by the time taken.

For the first part of the journey, $acceleration = 15 \div 10 = 1.5m/s^2$

For the second part of the journey, $acceleration = 0 \div 10 = 0$

For the third part of the journey, $acceleration = 10 \div 10 = 1m/s^2$

For the final part of the journey, $acceleration = -25 \div 20 = -1.25m/s^2$

Therefore, the maximum acceleration of the car was $1.5m/s^2$.

The steeper the line, the greater the acceleration

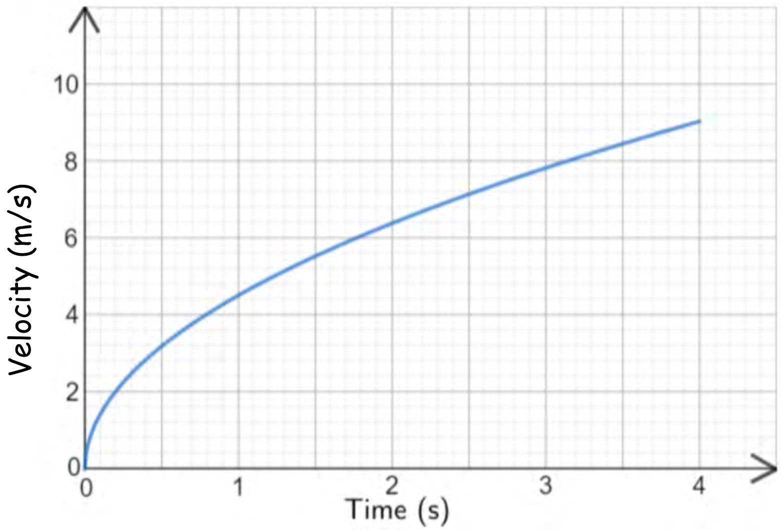
Mathematics

Higher

Unit 30

The gradient of a velocity time graph represents acceleration and deceleration.

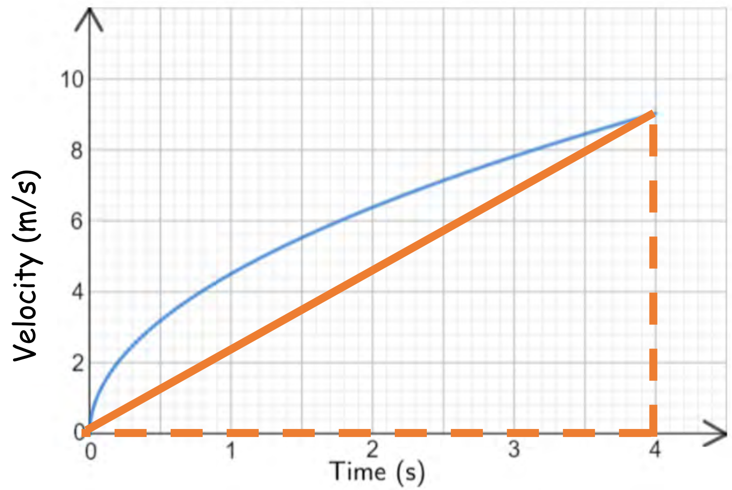
Example: Below is a speed-time graph of the first 4 seconds of someone running a race.



a) Work out the average acceleration over the 4 seconds.
b) Work out the instantaneous acceleration 2 seconds in.

a) To work out acceleration, work out the gradient.

To work out the average acceleration over 4 seconds, draw a line from 0s to 4s. Then find the gradient of that line by making it a right-angled triangle.



To find the gradient of the line we use the formula:

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

Where y_2 and y_1 are the coordinates on the y-axis, and x_2 and x_1 are the coordinates on the x-axis.

To find the gradient of th line, divide the y values by the x values.

For this graph, $y_2 = 9$, $y_1 = 0$, $x_2 = 4$, and $x_1 = 0$

$$gradient = \frac{9-0}{4-0} = 2.25m/s^2$$

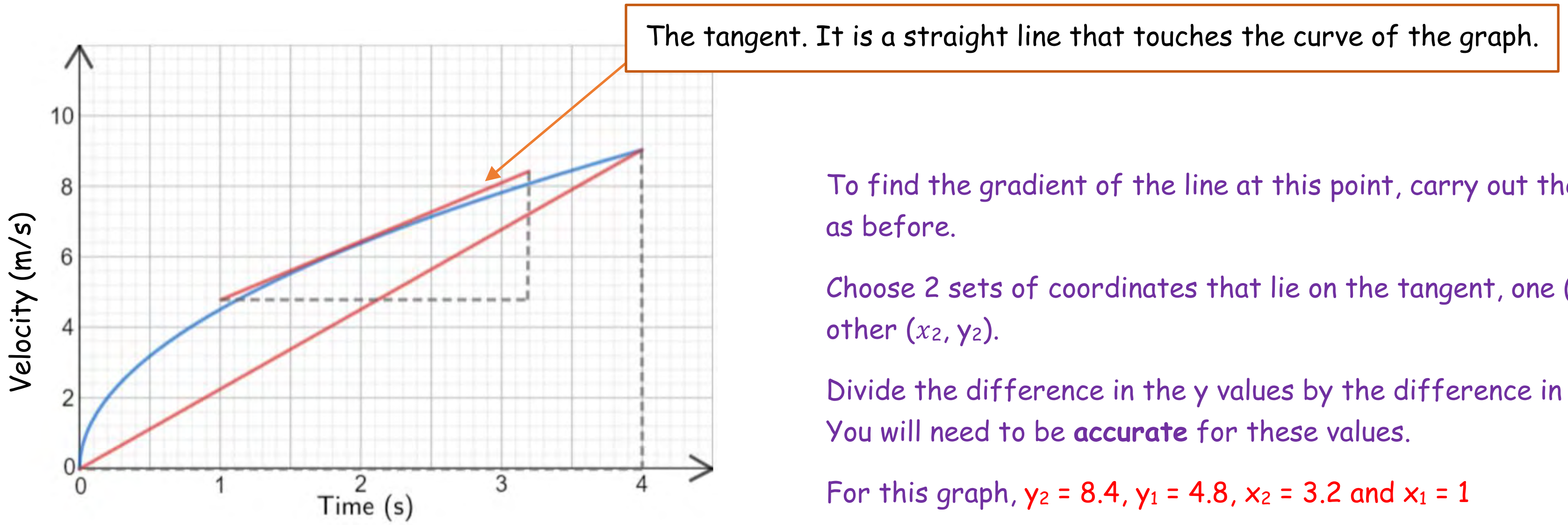
So, the average acceleration over the 4 seconds is $2.25m/s^2$

Mathematics

Higher

Unit 30

b) To work out the acceleration at 2s, draw a tangent to the line at 2 seconds and work out the gradient of that.



To find the gradient of the line at this point, carry out the same method as before.

Choose 2 sets of coordinates that lie on the tangent, one (x_1, y_1) and the other (x_2, y_2).

Divide the difference in the y values by the difference in the x values. You will need to be **accurate** for these values.

For this graph, $y_2 = 8.4$, $y_1 = 4.8$, $x_2 = 3.2$ and $x_1 = 1$

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

$$gradient = \frac{8.4 - 4.8}{3.2 - 1} = 1.64m/s^2$$

So, the instantaneous acceleration 2 seconds in is $1.64m/s^2$

Mathematics

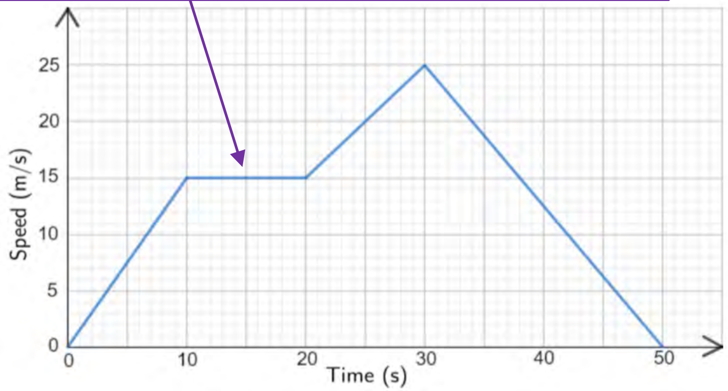
Higher

Unit 30

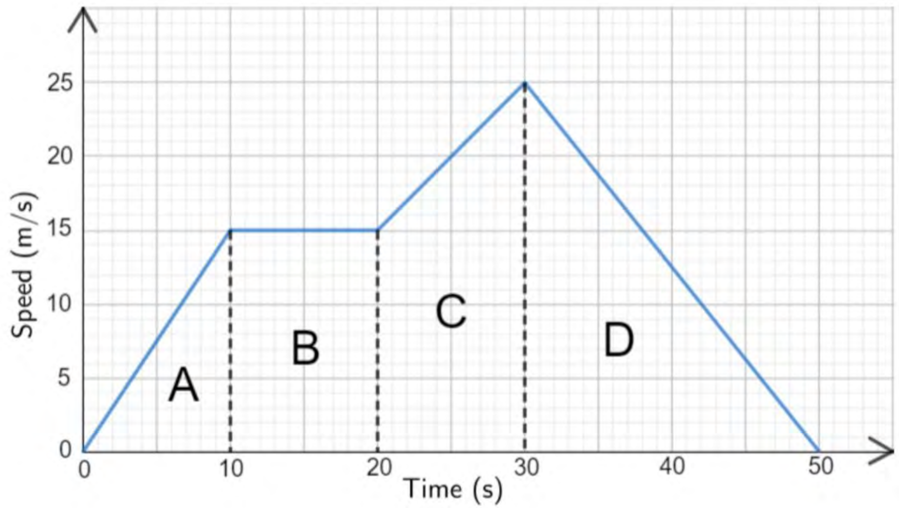
The area under a velocity time graph represents distance travelled.

Example 1: Work out the distance travelled from the graph below.

When the graph has a straight horizontal line, it means the acceleration is zero (travelling at a constant speed).



First, split the graph into sections. These sections are usually right-angled triangles, rectangles, and trapeziums.



Then find the areas of those shapes:

$Area\ A = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 10 \times 15$

$= 75m$

$Area\ B = b \times h$

$= 10 \times 15$

$= 150m$

$Area\ C = \frac{1}{2} \times (a + b) \times h$

$= \frac{1}{2} \times (15 + 25) \times 10$

$= 200m$

$Area\ D = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 20 \times 25$

$= 250m$

To get the total distance travelled, add up all the answers.

$Total\ distance\ travelled = 75m + 150m + 200m + 250m$

$= 675m$

Mathematics

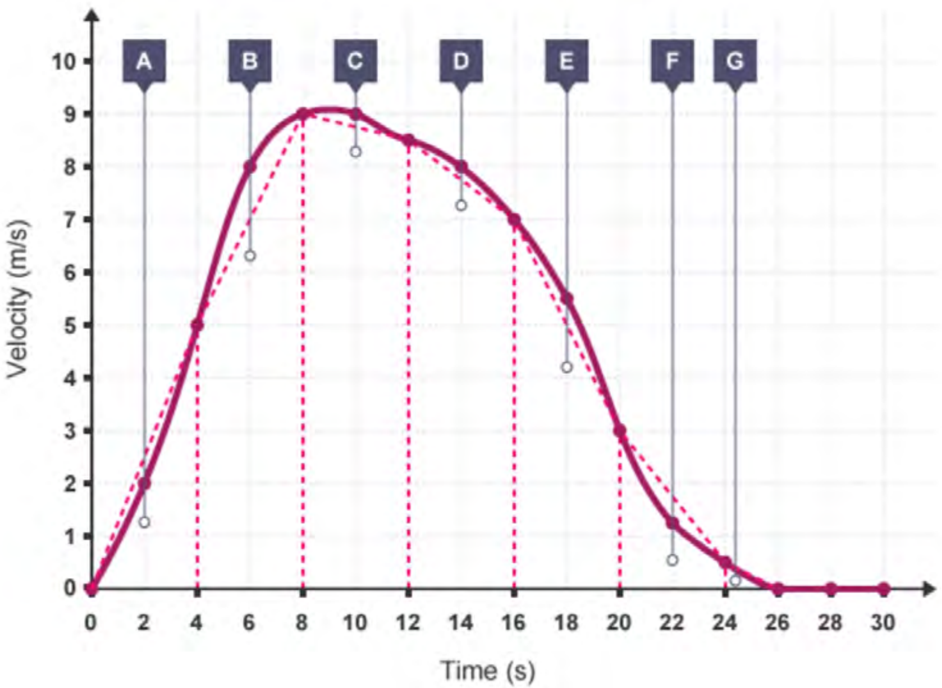
Higher

Unit 30

Sometimes we are given curved graphs and asked to find the distance travelled.

Example 2: Work out the distance travelled from the graph below.

To find the distance travelled from a curved graph, vertical lines are drawn along the horizontal axis. These vertical lines are connected to make triangles, or trapeziums that approximate to the curve.



Find the area of each of these sections.

$Area\ A = \frac{1}{2} \times b \times h$
$= \frac{1}{2} \times 4 \times 5$
$= 10m$

$Area\ B = \frac{1}{2} \times (a+b) \times h$
$= \frac{1}{2} \times (5+9) \times 4$
$= 28m$

$Area\ C = \frac{1}{2} \times (a+b) \times h$
$= \frac{1}{2} \times (9+8.5) \times 4$
$= 35m$

$Area\ D = \frac{1}{2} \times (a+b) \times h$
$= \frac{1}{2} \times (8.5+7) \times 4$
$= 31m$

$Area\ E = \frac{1}{2} \times (a+b) \times h$
$= \frac{1}{2} \times (7+3) \times 4$
$= 20m$

$Area\ F = \frac{1}{2} \times (a+b) \times h$
$= \frac{1}{2} \times (3+0.5) \times 4$
$= 7m$

$Area\ G = \frac{1}{2} \times b \times h$
$= \frac{1}{2} \times 2 \times 0.5$
$= 0.5m$

To get the total distance travelled, add up all the answers.

$Total\ distance\ travelled = 10m + 28m + 35m + 31m + 20m + 7m + 0.5m$
$= 131.5m$

Mathematics

Higher

Unit 31

Trigonometry

Just like Pythagoras' Theorem, Trigonometry only works with RIGHT-ANGLED TRIANGLES. However, Trigonometry can be used to find missing sides **or** missing angles.

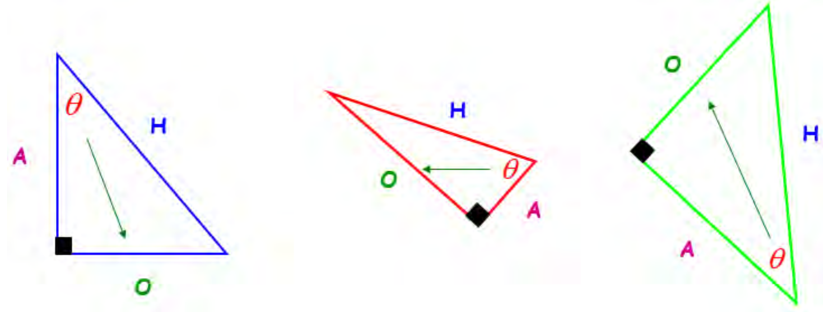
Labelling the Sides of a Right-Angled Triangle

This is the order to do it:

Hypotenuse (H) – the longest side, opposite the right-angle

Opposite (O) – the side directly opposite the angle you have been given / asked to work out

Adjacent (A) – the only side left.



Note: θ is just the Greek letter Theta, and it is used for unknown angles, just like x is often used for unknown lengths.

Sine, Cosine and Tangent (SOH CAH TOA)

Method for finding missing sides or angles:

Step 1: Label your right-angled triangle

Step 2: Tick which information (lengths of sides, sizes of angles) you have been given

Step 3: Tick which information you have been asked to work out

Step 4: Decide whether the question needs **sin**, **cos**, or **tan**

Learn the following formulas:

$$Sine\ \theta = \frac{Opposite}{Hypotenuse} \qquad Sin\ \theta = \frac{O}{H}$$

$$Cosine\ \theta = \frac{Adjacent}{Hypotenuse} \qquad Cos\ \theta = \frac{A}{H}$$

$$Tangent\ \theta = \frac{Opposite}{Adjacent} \qquad Tan\ \theta = \frac{O}{A}$$

Substitute in the two values you do know and re-arrange the equation to find the value you don't know.

A good way to remember the formulae is to use the initials from left to right:

SOH CAH TOA

To decide if you need to use sin, cos, or tan, you could highlight or tick the information you have been given and asked to work out.

Mathematics

Higher

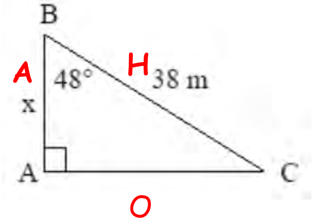
Unit 31

Solving Trigonometry Problems

When finding an angle, remember you need to use one of the inverse operations either sin^{-1}, cos^{-1} or tan^{-1}

You will need to press SHIFT on your calculator first.

Example 1: Find the length of AB (this can be labelled x)



Step 1: Label the sides O, A and H

Step 2 and 3: Tick or highlight what you have been given and what you have been asked to work out

SOH CAH TOA

Step 4: Decide whether you need sin, cos or tan. Looking above the only one that contains both A and H is **cos**

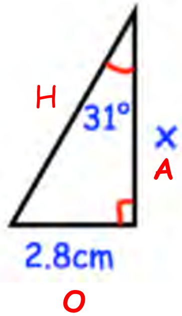
$$\cos\theta = \frac{A}{H}$$

$$\cos 48 = \frac{x}{38}$$

$$38 \times \cos 48 = x$$

$$x = 25.4m\ (1dp)$$

Example 2: Find the length of x



Step 1: Label the sides O, A and H

Step 2 and 3: Tick or highlight what you have been given and what you have been asked to work out

SOH CAH TOA

Step 4: Decide whether you need sin, cos, or tan. Looking above the only that contains both A and O is **tan**

$$\tan\theta = \frac{O}{A}$$

$$\tan 31 = \frac{2.8}{x}$$

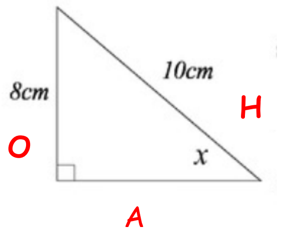
Note: Because the unknown x is on the bottom you would need to multiply up by x and divide by $\tan 31$ but it is easier just to swap them.

$$x = \frac{2.8}{tan31}$$

$$x = 4.66cm\ (2dp)$$

(on the calculator you can press 2.8 ÷ tan31 =)

Example 3: Find the size of the angle x



Step 1: Label the sides O, A and H

Step 2: Tick or highlight what you have been given

SOH CAH TOA

Step 4: Decide whether you need sin, cos, or tan. Looking above the only that contains both O and H is **sin**

$$\sin x = \frac{O}{H}$$

$$\sin x = \frac{8}{10}$$

(on the calculator you can press shift sin, 8 ÷ 10 =)

$$x = \sin^{-1}\left(\frac{8}{10}\right)$$

$$x = 53.1°\ (1dp)$$

Do not forget to round your answers to an appropriate degree of accuracy.

Mathematics

Higher

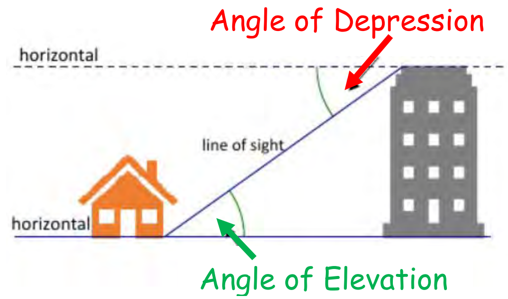
Unit 31

Solving multi-step Trigonometry problems

Some questions involve more than one step, so you may be required to use Trigonometry twice to find a missing angle and/or side. You may even have to use Pythagoras' Theorem in some questions like in example 3.

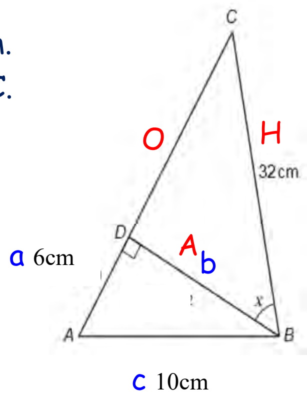
Some questions, like example 4, involve an angle of elevation or an angle of depression.

The Angle of Elevation is formed by looking UP from the horizontal, (stood at the house, looking up at the top of the tower).
The Angle of Depression is formed by looking DOWN from the horizontal, (stood on the top of the tower looking down at the house).



Example 3: AD = 6cm, AB = 10cm and BC = 32cm.
D is on the line AC and BD is perpendicular to AC.

Calculate the size of angle x to 1 decimal place.



Looking at the triangle BCD, you will need one more side before you can find angle x. The triangle ABD is right angled so Pythagoras' Theorem can be used to find BD.

$$b^2 = c^2 - a^2$$

$$= 10^2 - 6^2$$

$$= 100 - 36$$

$$b^2 = 64$$

$$So, \quad b = \sqrt{64} = 8$$

Therefore BD = 8cm and using Trigonometry angle x can be found.

$$\cos x = \frac{A}{H}$$

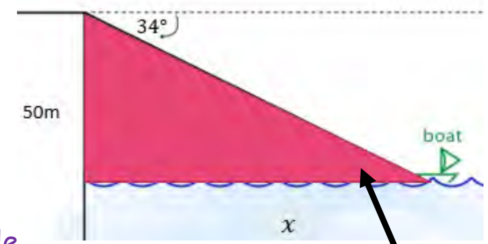
$$\cos x = \frac{8}{32}$$

$$x = \cos^{-1}\left(\frac{8}{32}\right)$$

$$x = 75.5^o \ (1dp)$$

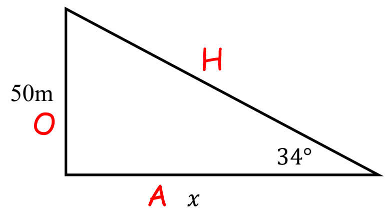
(on the calculator you can press shift cos, 8 ÷ 32 =)

Example 4: A man standing on a cliff sees a boat at an angle of depression of 34°. If the cliff is 50m tall, how far from the cliff is the boat?



Before we label the sides of the triangle, we need an angle inside the triangle. Using the alternate angles in parallel lines are equal rule, we can determine that this angle is also 34°.

Step 1: Label the sides O, A and H



Step 2 and 3: Tick or highlight what you have been given and what you have been asked to work out

SOH CAH TOA

Step 4: Decide whether you need sin, cos or tan. Looking above the only one that contains both O and A is **tan**

$$\tan\theta = \frac{O}{A}$$

$$\tan 34 = \frac{50}{x}$$

$$x = \frac{50}{tan34}$$

$$x = 74.13m \ (2dp)$$

Mathematics

Higher

Unit 31

Pythagoras & Trigonometry in 3D

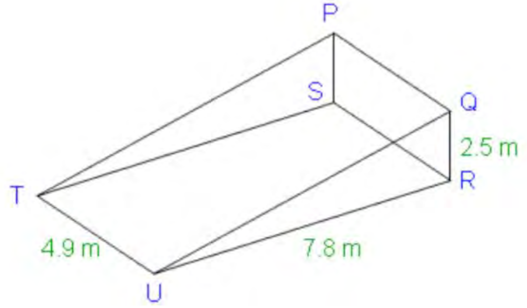
3D Trigonometry is just the same as normal trigonometry, the only difference is that it is more difficult to spot the right-angled triangles.

Once you spot them:

- Draw them out flat
- Label your sides
- Fill in the information that you do know
- Work out what you don't know in the usual way

Remember: You need a right-angled triangle to be able to use either Pythagoras or SOH CAH TOA.

Example 1: The diagram below shows a record-breaking wedge of Cheddar Cheese in which rectangle $PQRS$ is perpendicular (at 90°) to rectangle $RSTU$. The distances are shown on the diagram.



Calculate:

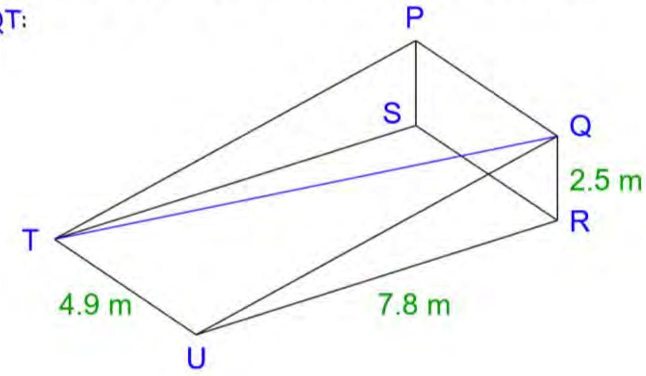
a) The distance QT

b) The angle $Q\hat{T}R$

Working out the answer, part a):

The first thing to figure out is what we are actually trying to work out.

We need the line QT:



The key to finding QT is spotting the right-angled triangle TQR.

It contains the length QT that we want, and we already know how long QR is.

However, we need to work out length TR first.

Mathematics

Higher

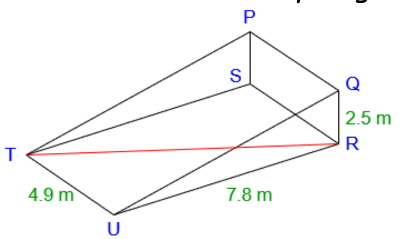
Unit 31

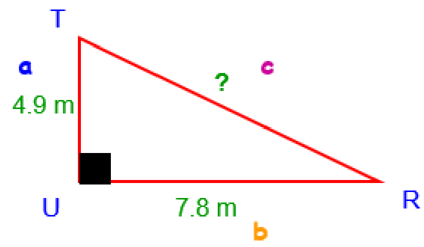
Working out the answer, part a) continued:

Working Out TR:

We can create a right-angled triangle (TRU) from the base rectangle $(RSTU)$.

We know two sides and we want to work out the Hypotenuse. This means we use Pythagoras' Theorem.





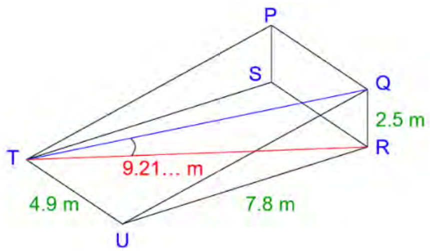
Hint
1. Label the sides
2. Use the formula:
3. Put in the numbers: $c^2 = a^2 + b^2$

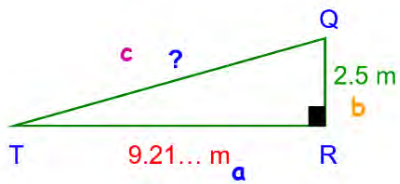
$$c^2 = 4.9^2 + 7.8^2$$
$$c^2 = 24.01 + 60.84$$
$$c^2 = 84.85$$
$$c = \sqrt{84.85}$$
$$c = 9.211...m$$

Working Out TQ:

For finding TQ we need to use the right-angled triangle QRT.

Once again, we know two sides and we want to work out the Hypotenuse. So, we use Pythagoras' Theorem.





Hint
1. Label the sides
2. Use the formula:
3. Put in the numbers: $c^2 = a^2 + b^2$

$$c^2 = 9.21...^2 + 2.5^2$$
$$c^2 = 84.85 + 6.25$$
$$c^2 = 91.1$$
$$c = \sqrt{91.1}$$
$$c = 9.54m \quad \text{(2dp)}$$

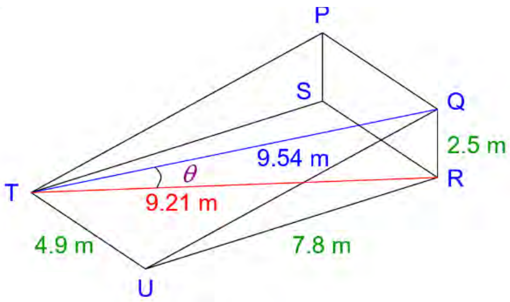
Mathematics

Higher

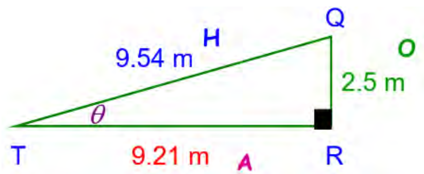
Unit 31

Working out the answer, part b):

Mark the angle QTR on the diagram



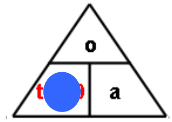
Draw the right-angled triangle:



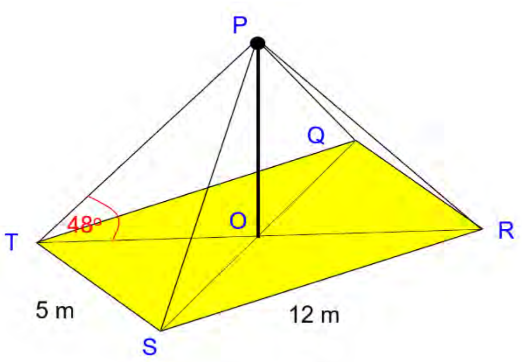
$$Tan\theta = o \div a$$
$$Tan\theta = 2.5 \div 9.21$$
$$Tan\theta = 0.27144\ldots$$
$$\theta = Tan^{-1}(0.27144\ldots)$$
$$\theta = 15.2° \ (1dp)$$

To calculate the size of an angle, we must use either **sin**, **cos** or **tan**, which means first we must label our sides!

Because we know all three lengths, we can choose what formula we want to use. In this case, Tan has been chosen.



Example 2: The diagram below shows a plan of a tent. OP is a vertical pole, and O is at the very centre of the rectangle $QRST$. The lengths and angles are as shown on the diagram. Calculate the height of the vertical pole OP.

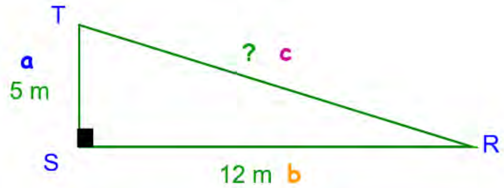


Working out OT:

You need to work out OT. You will then have a right-angled triangle which will give OP.

First use the base of the rectangle.

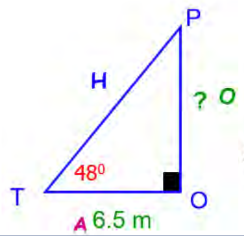
OT is half-way along the line TR. Use Pythagoras to find TR

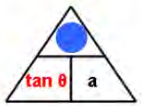


$$c^2 = 5^2 + 12^2$$
$$c^2 = 25 + 144$$
$$c^2 = 169$$
$$c = \sqrt{169}$$

Working out OP:

Now you have a right-angled triangle where we know one length, TR, and we know one angle, OTP. So, you can work out any side using SOH CAH TOA.





o = $Tan\theta \times a$

o = $(Tan48) \times 6.5$

o = 7.22m (2dp)

Mathematics

Higher

Unit 32

Transformations

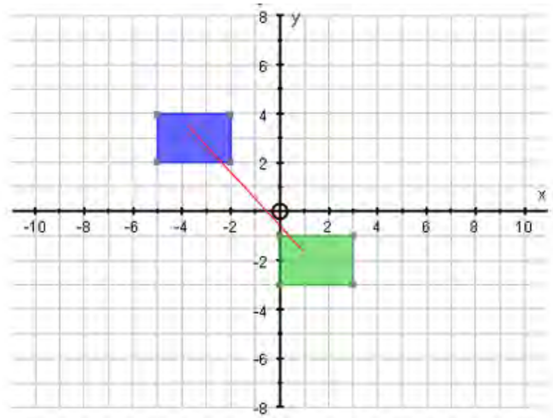
Transformations are specific ways of moving objects, usually around a co-ordinate grid

There are 4 types of transformations you need to know, translation, reflection, rotation, and enlargement, and for each one you must:

- be able to carry out a transformation yourself
- be able to describe a transformation giving all the required information

Translation

A Translation is a movement in a straight line, it is described by a movement right/left, followed by a movement up/down



If we translate the blue object 5 squares to the right and 5 squares down

We end up with the green object

Describing Translations

Translations can be described using words or vectors.

Example: Translate the object 2 squares to the right and 4 squares down.

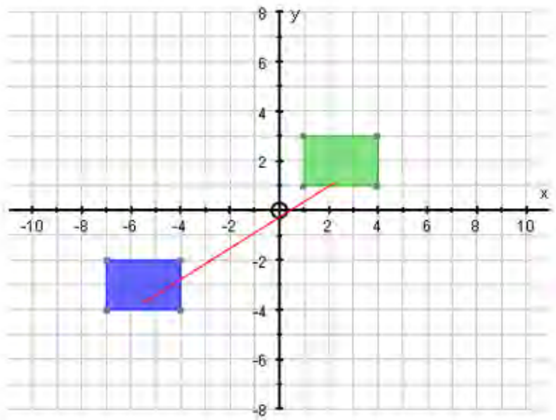
Or

Translate the object using the column vector $\binom{2}{-4}$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

If this number is positive you move right, if it is negative you move left.

If this number is positive you move up, if it is negative you move down.



If we translate the blue object by the vector: $\begin{pmatrix} 8 \\ 5 \end{pmatrix}$

8 to the right

5 up

We end up with the green object

Note: If you pick any co-ordinate on the blue shape and translate it by the same vector, you end up with the matching corner on the green shape

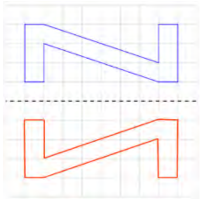
Mathematics

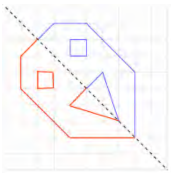
Higher

Unit 32

Reflection

Reflecting an object across a line produces an exact replica (mirror image) of that object on the other side of the line.



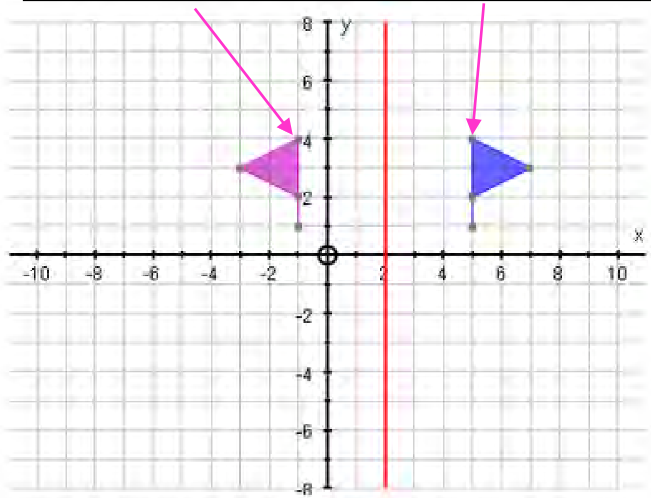


This new shape is called the Image

Describing Reflections

You must give either the equation of the line of reflection (mirror line) or draw the line on the grid.

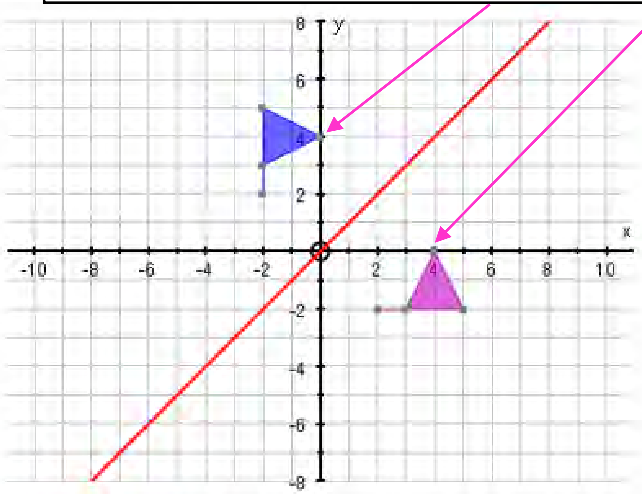
Note: This corner of the blue object is 3 squares from the line, so its corresponding point on the purple object will also be 3 squares from the



If we reflect the blue object in the red line (equation $x = 2$), we end up with the purple object

Note: Every point on the purple object (the image) is the exact same distance from the line of reflection as the matching point on the blue object

Note: This corner of the blue object is 4 squares horizontally from the line, as the line is diagonal its corresponding point on the purple object will be 4 squares vertically from the line



If we reflect the blue object in the red line (equation: $y = x$), we end up with the purple object

Note: Every point on the image is the same distance away from the mirror line as the matching point on the original object.

Mathematics

Higher

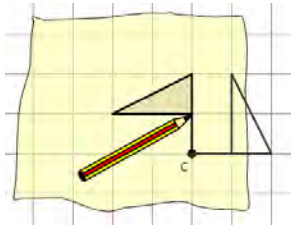
Unit 32

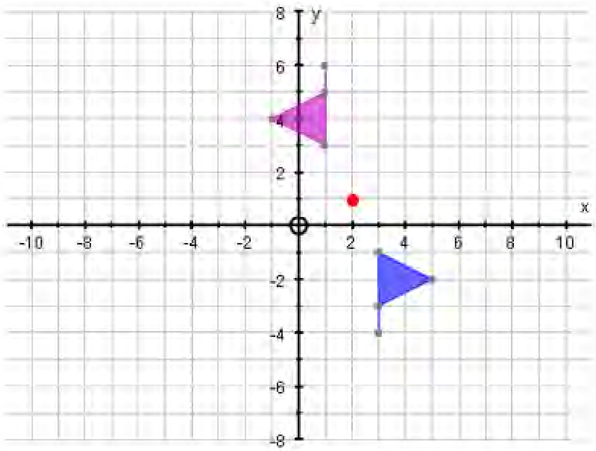
Rotation

Rotating an object means turning the whole shape around a fixed point by a certain number of degrees and in a certain direction.

Remember: If you cannot do these just by looking at the shape, then:

- trace around the object
- place your pencil at the centre of rotation (the fixed point)
- turn the tracing paper around
- draw your rotated object.





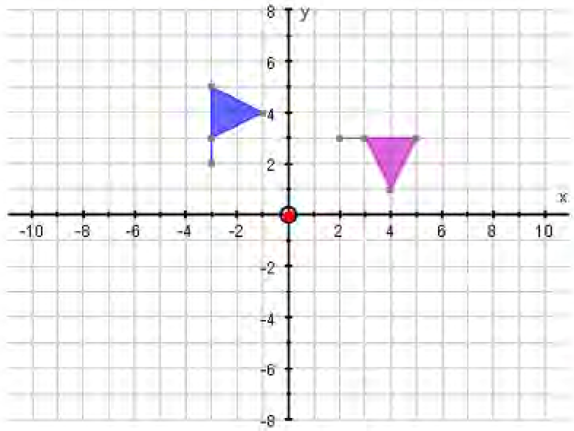
Rotating the blue object 180^0 about the point (2, 1) gives the purple object.

Note: Whenever the angle of rotation is 180^0, it doesn't matter whether you go clockwise or anti-clockwise.

Describing Rotations

You must give all of the following:

1. The **centre** of rotation (give as a co-ordinate if you can)
2. The **direction** of the rotation (clockwise or anti-clockwise)
3. The **angle** of the rotation (usually either 90^0, 180^0 or 270^0)



To describe the rotation from the blue object to the purple object, we would say:

1. Centre of Rotation: (0, 0) (the origin)
2. Direction of Rotation: Clockwise
3. Angle of Rotation: 90^0

Rotate the blue object 90^0 clockwise about the point (0, 0) (or about the origin)

Mathematics

Higher

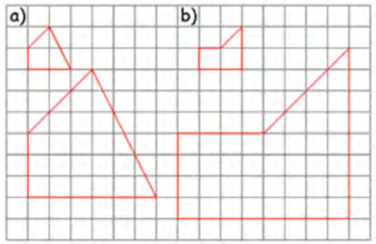
Unit 32

Enlargement

Enlargement is the only one of the four transformations which changes the size of the object

Note: Enlargements can make objects smaller as well as bigger

Each length is increased or decreased by the same scale factor



Going from small to big

(a) Scale Factor = 3

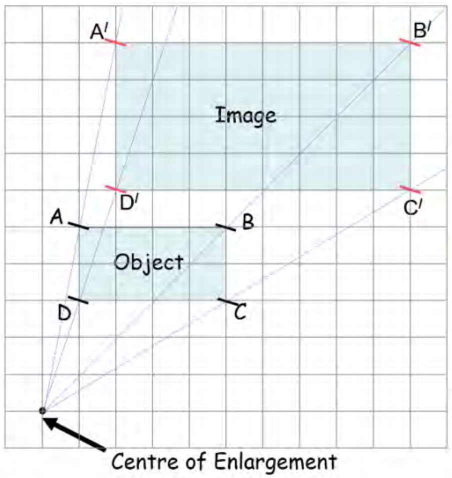
(b) Scale Factor = 4

And going from big to small

(a) Scale Factor = $\frac{1}{3}$

(b) Scale Factor = $\frac{1}{4}$

Enlargements from a Given Point



To enlarge the rectangle by scale factor x2 from the point shown

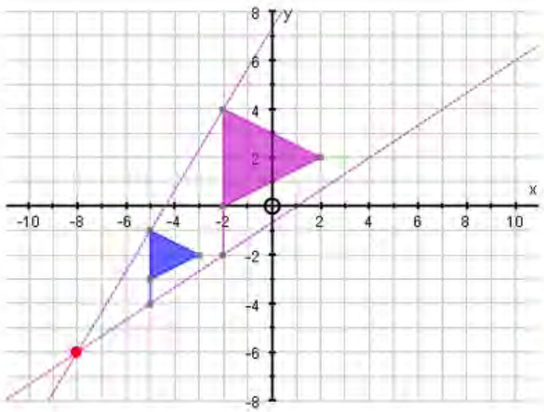
1. Draw the projection lines through corners.
2. Mark off x2 distances along each line.
3. Draw and label image.

Or Count Squares i.e. from the centre: A is 1 across, 5 up so A′ is going to be x2 so 2 across and 10 up.

Describing Enlargements

To fully describe an enlargement, you must give:

1. The centre of enlargement (give as a co-ordinate if you can)
2. The scale factor of the enlargement



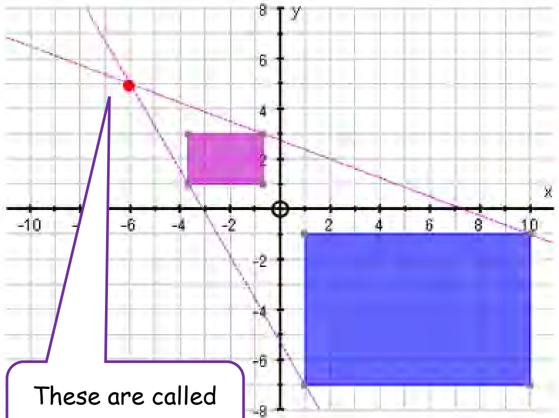
To describe the enlargement from the blue object to the purple object, we would say:

1. Centre of Enlargement: (-8, -6)
2. Scale Factor of Enlargement: 2

Note:

(1) To find the centre of enlargement you must draw line through matching points on both objects and see where they cross

(2) Each point on the purple object is twice as far away from the centre of enlargement than the matching point on the blue.



These are called projection lines.

To describe the enlargement from the blue object to the purple object, we would say:

1. Centre of Enlargement: (-6, 5)
2. Scale Factor of Enlargement: $\frac{1}{3}$

Note:

(1) The object has gone smaller, so it must be a fractional scale factor

(2) Each point on the purple object is one-third as far away from the centre of enlargement than the matching point on the blue.

Mathematics

Higher

Unit 32

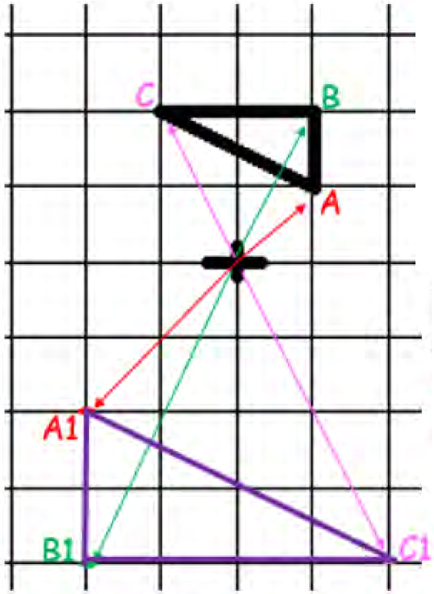
Negative Enlargements

A negative scale factor means that when we enlarge the shape, the shape ends up on the opposite side of the centre of enlargement. (It is like a twist and flip about the centre of enlargement).

Example 1: Enlarge triangle ABC by a scale factor of -2 using the cross as the centre of enlargement.

The shape will end up twice the size of the original shape but the other side of the centre of enlargement.

To get from the cross to C we go left 1 square and up 2 squares. Multiply this by 2 and go in the opposite direction. From the cross go right 2 squares and down 4 squares to make the new point (C1).



To get from the cross to B we go right 1 square and up 2 squares. Multiply this by 2 and go in the opposite direction. From the cross go left 2 squares and down 4 squares to make the new point (B1).

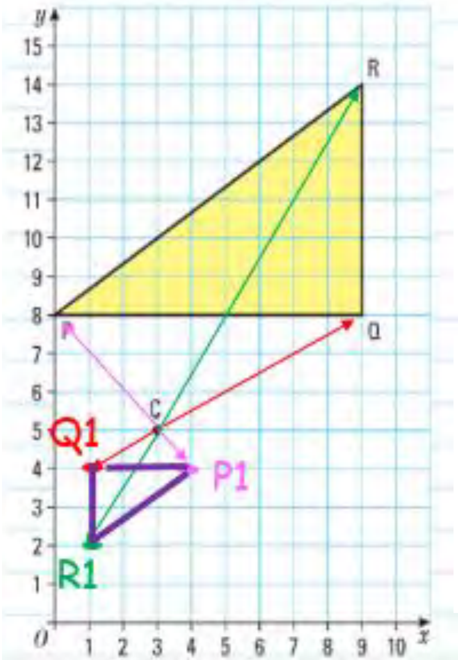
To get from the cross to A we go 1 diagonal square. Multiply this by 2. Go 2 diagonal squares in the opposite direction to make the new point (A1).

Then join the new points A1, B1 and C1 to make the new shape. The shape should be twice the size of the original shape and 'twisted and flipped'.

Example 2: Enlarge triangle PQR by a scale factor of $-\frac{1}{3}$ using point C as the centre of enlargement.

The shape will end up $\frac{1}{3}$ the size of the original shape but the other side of the centre of enlargement (C).

To get from the C to P, we go left 3 squares and up 3 squares. Divide these by 3 (find a third) and go in the opposite direction. From C, go right 1 square and down 1 square to make the new point (P1).



To get from C to R, we go right 6 squares and up 9 squares. Divide these by 3 (find a third) and go in the opposite direction. From C go left 2 squares and down 3 squares to make the new point (R1).

To get from the C to Q, we go right 6 squares and up 3 squares. Divide this by 3 (find a third) and go in the opposite direction. From C go 2 squares left and 1 square down to make the new point (Q1).

Then join the points to make the new shape. The shape should be $\frac{1}{3}$ the size of the original shape and 'twisted and flipped'.

Mathematics

Higher

Unit 33

Factorising

Factorising is the opposite of expanding brackets.
Factorising just means "putting back into brackets".

Factorising Using Common Factors

Method:

Step 1: Look for the highest **common factors** in each term (they could be letters or numbers)

Step 2: Place these common factors **outside** the bracket

Note: The key to successful factorising is understanding factors, and if it helps, why not just write down what each term means in full, it is easier then to spot the factors.

Example: $12a \longrightarrow 12 \times a$, $6y^2 \longrightarrow 6 \times y \times y$, $7pq^2 \longrightarrow 7 \times p \times q \times q$

Step 3: Write down what is now left inside the bracket – ask yourself: what do I need to multiply the term outside the bracket by to get my original term?

Step 4: Check carefully that there are **no more** common factors in your bracket

Step 5: **Check** your answer by expanding your brackets

Example 1: Factorise $7a + 21$

Step 1: Look for common factors in both numbers and letters:

Numbers: 7 and 21 $\longrightarrow$ Highest Factor = 7

Letters: there are no letters in the 2nd term, so we can't take any letters outside the bracket.

Step 2: We have:

$$7(\ ?\ +\ ?\ )$$

Step 3: Now we must figure out:

$$7 \times ? = 7a \longrightarrow a$$

$$7 \times ? = 21 \longrightarrow 3$$

Which gives us: $7(a + 3)$

Step 4: Check there are no more common factors left inside the bracket.

Step 5: Check the answer by expanding the brackets (on paper or in your head) to make sure you get the original question.

Mathematics

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Example 2: Factorise $10p + 15pq$

Step 1: Look for common factors in both numbers and letters:

Numbers: 10 and 15 → Highest Factor = 5

Letters: p and p q → Highest Factor = p

Step 2: We have:

$$5p\ (\ ?\ +\ ?\ )$$

Step 3: Now we must figure out:

$$5p \times ? = 10p \longrightarrow 2$$
$$5p \times ? = 15pq \longrightarrow 3q$$

Which gives us: $5p\ (2 + 3q)$

Step 4: Check there are no more common factors left inside the bracket.

Step 5: Check the answer by expanding the brackets (on paper or in your head) to make sure you get the original question.

Example 3: Factorise $24c^2 + 16c$

Step 1: Look for common factors in both numbers and letters:

Numbers: 24 and 16 → Highest Factor = 8

Letters: c^2 and c Highest Factor = c

Remember: c^2 is just c x c

Step 2: We have:

$$8c\ (\ ?\ +\ ?\ )$$

Step 3: Now we must figure out:

$$8c \times ? = 24c^2 \longrightarrow 3c$$
$$8c \times ? = 16c \longrightarrow 2$$

Which gives us: $8c(3c + 2)$

Step 4: Check there are no more common factors left inside the bracket.

Step 5: Check the answer by expanding the brackets (on paper or in your head) to make sure you get the original question.

Note: A very common mistake is not to take out the highest common factor.

For example, imagine we were doing **Example 3**, but for the numbers we thought the highest common factor was 2:

Numbers: 24 and 16 → Highest Factor = 2

Letters: c^2 and c → Highest Factor = c

We would get:

$$2c\ (\ ?\ +\ ?\ )$$

And then:

$$2c \times ? = 24c^2 \longrightarrow 12c$$
$$2c \times ? = 16c \longrightarrow 8$$

Which gives us: $2c(12c + 8)$

But, so long as we remember to always check there are no more common factors, we'll be fine, because a quick glance at this answers shows us that 12 and 8 have a common factor of 4.

Mathematics

Higher

Unit 33

Factorising Quadratic Expressions

Just like you had to expand double brackets, you also have to factorise quadratic expressions back into double brackets.

Quadratic expressions are algebraic expressions where the highest power of x is x

How to Factorise Quadratic Expressions

Factorising quadratics means you want to get from:

$$x^2 \boxed{\pm\ ?}x \boxed{\pm\ ?}$$

to

$$(\ x \boxed{\pm\ ?}\ )\ (\ x \boxed{\pm\ ?}\ )$$

$$(\ x \boxed{\pm\ ?}\ )\ (\ x \boxed{\pm\ ?}\ )$$

×

$$x^2 \boxed{\pm\ ?}x \boxed{\pm\ ?}$$

+

$$(\ x \boxed{\pm\ ?}\ )\ (\ x \boxed{\pm\ ?}\ )$$

Rules:

The two numbers (?) in the bracket (including their sign) must:

- Multiply together to give you the number (?) and sign at the end
- And add together to give you the number (?) and sign in front of the x

Example 1: $x^2 + 11x + 24$

$$x^2 \boxed{+\ 11}x \boxed{+\ 24}$$

We need to find two numbers which multiply together to give 24 and add together to give 11.

You might find it helpful to write down all the pairs of numbers which multiply together to give 24, and see which one also adds up to 11

1 x 24	1 + 24 = 25
2 x 12	2 + 12 = 14
3 x 8	3 + 8 = 11

Once we have our pair, we can just write the numbers in the brackets, remember that no sign means a plus

$$(x + 3)(x + 8) \text{ or } (x + 8)(x + 3)$$

Check: expand the brackets to make sure you are correct.

Example 2: $p^2 + 2p - 15$

$$p^2 \boxed{+\ 2}p \boxed{-\ 15}$$

We need to find two numbers which multiply together to give -15 and add together to give 2.

Write down pairs of numbers that multiply together to give -15, and see which one also adds up to 2

1 x -15	1 + -15 = -14
-1 x 15	-1 + 15 = 14
3 x -5	3 + -5 = -2
-3 x 5	-3 + 5 = 2

Once we have our pair, we can just write the numbers in the brackets, making sure we get our signs in the correct place.

$$(p - 3)(p + 5) \text{ or } (p + 5)(p - 3)$$

Check: expand the brackets to make sure you are correct.

Example 3: $k^2 - 13k - 14$

$$k^2 \boxed{-\ 13}k \boxed{-\ 14}$$

We need to find two numbers which multiply together to give -14 and add together to give -13.

Write down pairs of numbers that multiply together to give -14, and see which one also adds up to -13

-1 x 14	-1 + 14 = 13
1 x -14	1 + -14 = -13

Once we have our pair, we can just write the numbers in the brackets, making sure we get our signs in the correct place.

$$(k + 1)(k - 14) \text{ or } (k - 14)(k + 1)$$

Check: expand the brackets to make sure you are correct.

Example 4: $v^2 - 9v + 18$

$$v^2 \boxed{-\ 9}v \boxed{+\ 18}$$

We need to find two numbers which multiply together to give 18 and add together to give -9.

Write down pairs of numbers that multiply together to give 18, and see which one also adds up to -9 (We need two negative numbers)

-1 x -18	-1 + -18 = -19
-2 x -9	-2 + -9 = -11
-3 x -6	-3 + -6 = -9

Once we have our pair, we can just write the numbers in the brackets, making sure we get our signs in the correct place.

$$(v - 3)(v - 6) \text{ or } (v - 6)(v - 3)$$

Check: expand the brackets to make sure you are correct.

Mathematics

Higher

Unit 33

Factorising - the Difference of Two Squares

This question looks a little different to the previous ones.

It does not have an 'x' term. But the same question can be asked.

$$x^2 - 16$$

Which two numbers multiply together to give -16 and add together to give 0?

Remember, it is the number in front of the x which tells you what the numbers must add together to make, but as there are no x's, the sum (total) of the two numbers must be 0!

Think of the expression as $x^2 + 0x - 16$

For two numbers to add together to give zero, they must be the same number, but have opposite signs so that they cancel each other out!

The two numbers needed are +4 and -4! Expand it to check.

$$x^2 - 16 \longrightarrow (x + 4)(x - 4)$$

Expressions like this are called "the difference of two squares", and are always factorised in a similar way.

$$a^2 - 25 \longrightarrow (a + 5)(a - 5)$$

$$p^2 - 100 \longrightarrow (p + 10)(p - 10)$$

$$4t^2 - 49 \longrightarrow (2t + 7)(2t - 7)$$

Method:

Square root both terms.

Put the same answers in each bracket BUT one sign will always be a '+' and the other sign will always be a '-'.

Note: The question must always have a minus sign between the 2 terms.

Both terms must be squared terms.

For more difficult questions remember that the square root of values such as x^6 is x^3.

Mathematics

Higher

Unit 33

Factorising Harder Quadratics

Harder quadratics are quadratic expressions where the number in front of x^2 is bigger than 1. In other words, there is a number in front of the squared term.

For example, $5g^2 + 6g - 27$ OR $6c^2 - 23c + 20$

Example 1: Factorise $2x^2 - x - 3$

This time, instead of looking for two numbers which multiply together to give -3, we need to look for two numbers which multiply together to give $2 \times -3 = -6$ (the number in front of the x^2, times the number on its own) and add together to give -1 (the number in front of the x).

$2x^2 - x - 3$

Multiply to give 2 x -3 = -6

$2x^2 - x - 3$

Add to give -1

The pairs of numbers which multiply to give -6 are:

1 x -6

-1 x 6

2 x -3

-2 x 3

This pair work as + 2 -3 = -1

Normally, we put them into brackets as $(x + 2)(x - 3)$ BUT this time there was a 2 in front of the x^2.
So, we start with $(2x + 2)(2x - 3)$ and then check to see if we can divide any of the brackets.

$(2x + 2)(2x - 3)$

The first bracket can be divided by 2.

$\frac{(2x+2)}{2}(2x - 3)$

Dividing $(2x + 2)$ by 2 gives $(x + 1)$

$= (x + 1)(2x - 3)$

Mathematics

Higher

Unit 33

Example 2: Factorise $8x^2 - 22x + 15$

This time, we need to look for two numbers which multiply together to give $8 \times 15 = 120$ (the number in front of the x^2, times the number on its own) and add together to give -22 (the number in front of the x).

$8x^2 - 22x + 15$

Multiply to give 8 x 15 = 120

$8x^2 - 22x + 15$

Add to give -22

We must be looking for two negative numbers as the second sign is + telling us that the signs are the same and both - as the first is a -.

Normally, we put them into brackets as $(x - 10)(x - 12)$ BUT this time there was an 8 in front of the x^2.

So, we start with $(8x - 10)(8x - 12)$ and then check to see if we can divide any of the brackets.

The pairs of numbers which multiply to give 120 are:

-1 x -120

-2 x -60

-3 x -40

-4 x -30

-5 x -24

-6 x -20

-8 x -15

-10 x -12

This pair work as -10 + -12 = -22

$(8x - 10)(8x - 12)$

$\frac{(8x-10)}{2} \quad \frac{(8x-12)}{4}$

$= (4x - 5)(2x - 3)$

The first bracket can be divided by 2 and the second bracket can be divided by 4.

Mathematics

Higher

Unit 34

Solving Quadratic Equations

There are two ways to solve quadratic equations for GCSE.

1. Factorising
2. Using the Quadratic Formula.

Which ever way you solve a quadratic equation, you must remember the Golden Rule

The Golden Rule for Solving Quadratic Equations: You should always get TWO answers.
This is because quadratics contain squares, and what happens when you square negative numbers.
For example: $x^2 = 25$
A solution is $x = 5$ but you can also have $x = -5$.
This is because when you square a negative number, you get a positive answer.
That is why there are two solutions when quadratics are involved.

1. Solving by Factorising

Note: You need to be able to factorise in order to solve quadratic equations.

Method

1. Rearrange the equation to make it equal to zero
2. Factorise the quadratic equation
3. Put each bracket equal to zero separately
4. Solve each new equation separately to give two answers.

Why does it work?

After following steps 1. and 2, you may have the following 2 brackets equal to zero:

$$(x - 4)(x + 3) = 0$$

This means you have two things (x - 4) and (x + 3) that when multiplied together (disguised multiplication sign between the brackets) equal zero.
If two things multiplied together equal zero it means at least one of them must be zero.
You need to ask yourself: "what value of x makes the first bracket equal to zero?" (= 4)
And "what value of x makes the second bracket equal to zero?" (-3)

This means that the 2 answers for x are:

$x = 4$ or $x = -3$

Example 1

$$x^2 - 3x - 28 = 0$$

Using the method:

1. The equation is already equal to zero
2. Factorise the left hand side

$$x^2 - 3x - 28 \rightarrow (x - 7)(x + 4)$$

In terms of the equation, you have:

$$(x - 7)(x + 4) = 0$$

3. Put each bracket equal to zero. We have:

(x - 7) = 0 OR (x + 4) = 0

4. Solve each new equation separately.

$x - 7 = 0$		$x + 4 = 0$
$x = 0 + 7$		$x = 0 - 4$
$x = 7$	OR	$x = -4$

Mathematics

Higher

Unit 34

Solving Quadratic Equations

Example 2

$$2x^2 + 5x = 3$$

1. This equation is NOT equal to zero but if we subtract 3 from both sides OR move the 3 to the right hand side and change it to minus 3, the equation is then equal to zero.

$$2x^2 + 5x - 3 = 0$$

2. This is a harder quadratic to factorise:

$$2x^2 + 5x - 3 \rightarrow (2x - 1)(x + 3)$$

In terms of the equation, we have:

$$(2x - 1)(x + 3) = 0$$

3. Put each bracket equal to zero. We have:

$(2x - 1) = 0$ OR $(x + 3) = 0$

4. Solve each new equation separately.

$2x - 1 = 0$ | $x + 3 = 0$

$2x = 1$ | $x = 0 - 3$

$x = \frac{1}{2}$ OR $x = -3$

Example 3

$$x^2 - 49 = 0$$

1. The equation is already equal to zero.

2. Factorise: this is factorising the difference of 2 squares

$$x^2 - 49 \rightarrow (x + 7)(x - 7) = 0$$

3. Put each bracket equal to zero.

$(x + 7) = 0$ OR $(x - 7) = 0$

4. Solve each new equation separately.

$x + 7 = 0$ | $x - 7 = 0$

$x = 0 - 7$ | $x = 0 + 7$

$x = -7$ OR $x = 4$

Example 4

$$x^2 + 4x = 0$$

1. The equation is already equal to zero.

2. Factorise: this is factorising using common factors.

$$x^2 + 4x \rightarrow x(x + 4) = 0$$

Think of the x outside the bracket as an x in a bracket on its own $(x)(x + 4) = 0$

3. Put each bracket equal to zero.

$x = 0$ OR $(x + 4) = 0$

4. Solve each new equation separately.

$x = 0$ | $x + 4 = 0$

$x = 0$ | $x = 0 - 4$

$x = 0$ OR $x = -4$

Mathematics

Higher

Unit 34

2. Solving by using the Quadratic Formula

The quadratic formula can solve every single quadratic equation. It looks more complicated than it actually is. You generally use the formula method on a calculator paper.

The Quadratic Formula:

+ or – this is where the 2 answers come from

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: To be able to use this formula, you must use you calculator carefully. Always double check your calculator workings.

What do the letters stand for?

The letters are the coefficients (the numbers in front) of the unknowns in your equation. Remember: you **must** include the signs of the numbers as well.

$$ax^2 + bx + c = 0$$

This is known as the general form of a quadratic.

Example

$$5x^2 - 8x + 12 = 0 \longrightarrow a = 5 \quad b = -8 \quad c = 12$$

Remember: Before you start putting numbers into the formula, you must make sure that you rearrange your equation to make it equal to zero.

HINT: It helps to put negative numbers in brackets.

Example 1

$$x^2 - 4x + 2 = 0$$

This equation cannot be solved using factorisation. This means that the formula will need to be used.

The equations is already equal to zero, so find the values of a, b and c.

$$ax^2 + bx + c = 0$$

$$x^2 - 4x + 2 = 0$$

$$a = 1 \quad b = -4 \quad c = 2$$

Note: a = 1, and not 0
Remember, the 1 is hidden.

Substitute the numbers in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

Since b = -4, -b = - -4. Two –'s meeting make a + so –b is 4.

$$x = \frac{4 \pm \sqrt{8}}{2}$$

Now, we can split the equation to give two separate answers.

$$x = \frac{4 + \sqrt{8}}{2} \quad \text{OR} \quad x = \frac{4 - \sqrt{8}}{2}$$

Now, type these into your calculator separately.

$x = 3.414213 \ldots$ OR $x = 0.585786 \ldots$

and round to something sensible (or what the questions asks you to round to).

x = 3.41 OR x = 0.59

Mathematics

Higher

Unit 35

Trial and Improvement

Trial and improvement is a method for finding an *approximate* solution to an equation that you would not be able to solve using your normal way.

It will tell you in the question when to use trial and improvement.

The idea of trial and improvement is to keep trying different values of x to get you closer and closer to the solution. If you remember the method, then they are easy marks to pick up.

Method

1. Draw a table for your values (*see example*).
2. Substitute **two** values into the equation: These two values are usually given to you in the question and will result in an answer that is too big and one that is too small i.e. opposite cases.
3. Substitute the next value into the equation which is between the previous two values: Choose the middle value (or close to the middle) of what you've already substituted.
4. If the last value is too small substitute in the next higher consecutive value correct to 1.d.p into the equation. If the last value is too big substitute in the next lower consecutive value correct to 1.d.p into the equation.
5. Keep going until you have two values next to each other in which one is too big, and one is too small.
6. **Substitute in the value** halfway between these two numbers.
   If the halfway answer is too small choose the bigger value correct to 1.d.p
   If the halfway answer is too big choose the smaller value correct to 1.d.p
   E.g. for 2.3 and 2.4 you would try 2.35 to see if the correct answer was between 2.3 and 2.35 (so you'd give 2.3 to 1dp) or between 2.35 and 2.4 (hence you'd give 2.4 as your answer to 1dp).

Example

The solution to the equation $x^3 + 9x = 40$ lies between 2 and 3.

Use trial and improvement to find the solution correct to 1dp.

Values from the question

x	$x^3 + 9x$	**Comment**
2	$2^3 + 9(2) = 26$	Too small
3	$3^3 + 9(3) = 54$	Too big
2.5	$2.5^3 + 9(2.5) = 38.125$	Too small
2.6	$2.6^3 + 9(2.6) = 40.976$	Too big
2.55	$2.55^3 + 9(2.55) = 39.531375$	Too small

This means x is between 2.55 and 2.6

$\therefore x = 2.6$

Mathematics

Higher

Unit 36

Changing the Subject of a Formula

Changing the subject of a formula means to use inverse operations to isolate a specified variable.
Example: Make x the subject of the formula $x + 7y = 12$ means we need to get $x =$ something as the answer, in this case it would be $x = 12 - 7y$

Examples (using the "change side, change sign" method of solving equations)

We keep moving terms over the equals sign until we get the specified letter on its own.

Example 1:

Rearrange $3m = 4b - 18$ to make b the subject of the formula

$3m = 4b - 18$

$3m + 18 = 4b$

$\frac{3m+18}{4} = b$

Example 2:

Rearrange $Q = \frac{2r-7}{3}$ to make r the subject of the formula:

$Q = \frac{2r-7}{3}$

$3Q = 2r - 7$

$3Q + 7 = 2r$

$\frac{3Q+7}{2} = r$

Example 3:

Rearrange $2(3a - c) = 5c + 1$ to make c the subject of the formula:

$2(3a - c) = 5c + 1$

expand brackets

$6a - 2c = 5c + 1$

$6a = 5c + 1 + 2c$

$6a = 7c + 1$

$6a - 1 = 7c$

$\frac{6a-1}{7} = c$

Example 4:

Rearrange $\sqrt{4a + r} = 3t$ to make r the subject of the formula:

$\sqrt{4a + r} = 3t$

The square root moves over the other side and turns into the opposite / inverse, which is a squared

$4a + r = (3t)^2$

$4a + r = 9t^2$

$r = 9t^2 - 4a$

Example 5:

Rearrange $38m = 7b^2 + 11$ to make b the subject of the formula:

$38m = 7b^2 + 11$

$38m - 11 = 7b^2$

$\frac{38m - 11}{7} = b^2$

The squared sign moves over the other side and turns into the opposite / inverse, which is a square root

$\sqrt{\frac{38m - 11}{7}} = b$

Mathematics

Higher

Unit 36

Examples (using the "balance" method of solving equations)

We undo each operation starting from the last one until we get the specified letter on its own.

Example 1:

Rearrange $3m = 4b - 18$ to make *b* the subject of the formula

$$3m = 4b - 18$$

+18 +18

$$3m + 18 = 4b$$

÷ 4 ÷ 4

$$\frac{3m+18}{4} = b$$

Example 2:

Rearrange $Q = \frac{2r-7}{3}$ to make *r* the subject of the formula:

$$Q = \frac{2r-7}{3}$$

×3 × 3

$$3Q = 2r - 7$$

+7 +7

$$3Q + 7 = 2r$$

÷ 2 ÷ 2

$$\frac{3Q+7}{2} = r$$

Example 3:

Rearrange $2(3a - c) = 5c + 1$ to make *c* the subject of the formula:

$$2(3a - c) = 5c + 1$$

expand

$$6a - 2c = 5c + 1$$

+2c +2c

$$6a = 7c + 1$$

-1 -1

$$6a - 1 = 7c$$

÷ 7 ÷ 7

$$\frac{6a - 1}{7} = c$$

Example 4:

Rearrange $\sqrt{4a + r} = 3t$ to make r the subject of the formula:

$$\sqrt{4a + r} = 3t$$

square square

$$4a + r = 9t^2$$

$-4a$ $-4a$

$$r = 9t^2 - 4a$$

Example 5:

Rearrange $38m = 7b^2 + 11$ to make *b* the subject of the formula:

$$38m = 7b^2 + 11$$

-11 -11

$$38m - 11 = 7b^2$$

÷ 7 ÷ 7

$$\frac{38m - 11}{7} = b^2$$

√ √

$$\sqrt{\frac{38m - 11}{7}} = b$$

Mathematics

Higher

Unit 37

Changing the Subject of the Formula (When the Subject Appears Twice)

Sometimes you will see a question like this:

Make x the subject: a) $4 = \frac{7y + 3x}{xy}$ b) $cx - g = ix + n$

Remember: The question will not always ask you to make x the subject of the formula. It could be different letters.

We are being asked to make x the subject of the formula. In other words, we want to rearrange this formula so our answer is in the form x = something (there will be letters in our answer).

If you look at these examples, you can see that x appears MORE THAN ONCE.

Example 1: Make x the subject $4 = \frac{7y + 3x}{xy}$

$4xy = 7y + 3x$ Multiply by xy to get rid of the fraction (we do not like there being an x as part of the denominator)

$4xy - 3x = 7y$ Subtract $3x$ to get all the terms with an x on the left-hand side of the ' = '

$x(4y - 3) = 7y$ Factorise the left-hand side (we now only have one x)

$x = \frac{7y}{4y - 3}$ Divide by the bracket to get x on its own. We can now leave out the bracket.

Method:

Step 1: If there is a denominator, this needs to be dealt with first. Sometimes, brackets will need to be expanded.

Step 2: Rearrange the formula so that all of the terms with an x in are on the left-hand side and all terms without an x are on the right-hand side.

Step 3: Factorise the left-hand side, taking x outside the bracket as a common factor.

Step 4: Divide by the bracket to get x on its own.

Example 2: Make x the subject $cx - g = ix + n$

$cx - ix - g = n$ Subtract ix to get all the terms with an x on the left-hand side of the ' = ' sign

$cx - ix = n + g$ Add g to get all the terms without an x on the right-hand side of the ' = '

$x(c - i) = n + g$ Factorise the left-hand side (we now only have one x)

$x = \frac{n + g}{c - i}$ Divide by the bracket to get x on its own. We can now leave out the bracket.

Mathematics

Higher

Unit 37

Pie Charts

Pie charts use angles to represent proportionally the quantity of each group involved.

Pie charts can only be compared to one another when populations are given.

Big Example 1:

A group of 72 maths teachers were asked to choose their favourite TV show from a list, and their responses are shown in the table on the right. Construct a pie chart to illustrate this information.

TV Show	Total
Lost	12
Heroes	10
Desperate Housewives	4
Countdown	15
Teachers TV	13
The Beauty of Maths	18

Working out the Angles

- Before you can start to draw the pie chart, you need to know how big a slice each of the choices is going to take up – in other words, you need to know the angle of each segment
- To work this out, you need to remember that there are 360 degrees in a circle
- That means there are 360 degrees to share between each of the people who took part in the

We have a total of 72 teachers who were surveyed. 360 ÷ 72 = 5

Each teacher is worth 5 degrees on our pie chart.

We now need to work out what angle each segment (each TV show) gets.

TV Show	Total	Working Out	Angle of Segment
Lost	12	12 x 5 = 60	60°
Heroes	10	10 x 5 = 50	50°
Desperate Housewives	4	4 x 5 = 20	20°
Countdown	15	15 x 5 = 75	75°
Teachers TV	13	13 x 5 = 65	65°
The Beauty of Maths	18	18 x 5 = 90	90°

Remember: Check this column adds up to 360 before you move on.

It is also worth noting that the overall total may NOT be a factor of 360.

72 is a factor of 360 (it goes into perfectly) so we just had to multiply by 5 to get each angle.

If the total is not a factor of 360, try using this method for working out the angles…

$Lost\ \frac{12}{72} \times 360 = 60^o$ and $Heroes\ \frac{10}{72} \times 360 = 50^o$

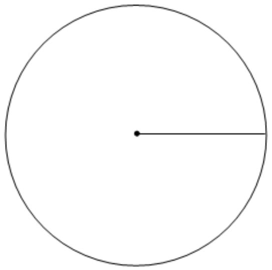
Mathematics

Higher

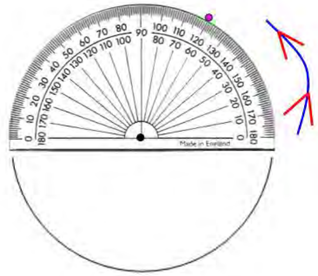
Unit 37

Drawing the Pie Chart

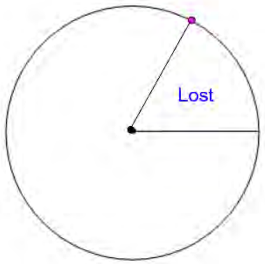
1. Draw a circle using a compass. Mark the centre with a dot and draw a straight line from the centre up to the right of your circle



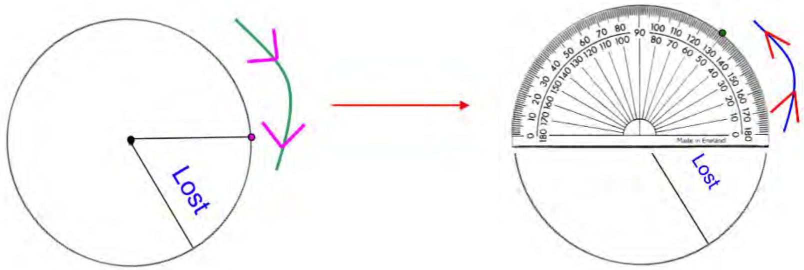
2. Carefully place your angle measurer along the line, with the **centre exactly on the centre of the circle.** Now, count around from 0 until you reach the correct number of degrees – in this case 60° – and place a dot



3. Join up your dot to the centre with a straight line and label your segment.

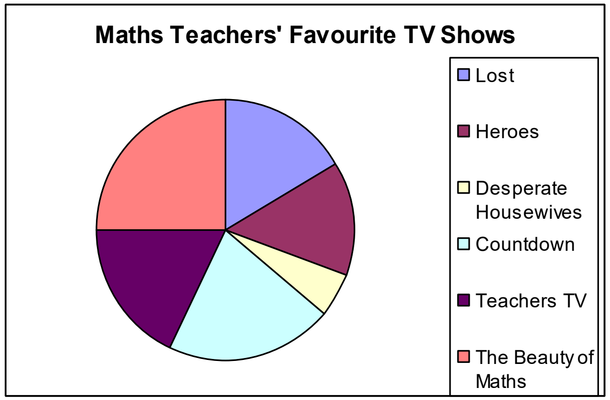


4. Turn your pie chart clockwise until your new line is horizontal (where the first line used to be). Now you can mark your next angle in the same way.



5. Keep doing this until you have drawn all your segments

6. If you want to you can colour in your segments, but you must remember to label them clearly, or add a key.



Check: You will know if you have got it right if the line to make your final segment is the very first line you drew.

Mathematics

Higher

Unit 37

What CAN we tell from Pie Charts?

If you look back at the pie chart in the last example, you will see The Beauty of Maths was the most popular choice amongst our maths teachers, whereas Desperate Housewives was the least popular.

You could also say something like "roughly 3 times as many teachers preferred Lost to Desperate Housewives".

What CAN'T we tell from Pie Charts?

If we were just given the pie chart (and no original data), and were asked "how many maths teachers said that Countdown was their favourite show?", there would be no way of knowing what the answer was.

Unless we are told how many people were surveyed all together, we cannot answer that question.

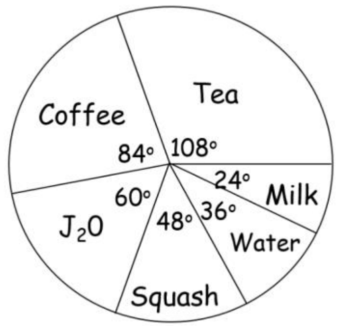
When making statements based on Pie Charts, just make sure what you are saying is definitely, 100% true.

Interpreting Pie Charts

Big Example 2:

240 Maths teachers were asked "what is your favourite drink?" and the pie chart on the right was drawn to show the information.

Work out how many teachers preferred coffee.



To answer this question, we must do the opposite of what we did when we were drawing the pie chart – we must use our angles to find our totals.

Let us look at the coffee segment, it takes up 84° out of 360°, and what we want to know is "how much does it take up out of our 240 people?"

$$\frac{84}{360} = \frac{?}{240} \xrightarrow{\text{Multiply both sides by 240}} \frac{84}{360} \times 240 = ?$$

Typing this in on the calculator gives an answer of 56 people

Sometimes the angles will not be given so you would have to use a protractor to measure each section.

If a percentage was given instead of an angle, for example 30% preferred Tea, to work out how many teachers this is, you would use a similar method.

$\frac{30}{100} \times 240 = 72$ (Use 100 instead of 360 because percentages are out of 100)

Mathematics

Higher

Unit 38

Questionnaires

Questionnaires or surveys are used to gather data.
You will be required to design or criticise questions on questionnaires.

What to look out for in questionnaires:

Overlapping interval

☐ 0 – 1 hours ☐ 1 – 2 hours ☐ 3 – 4 hours

Which box do you tick if your answer is one hour?

No time scale given

How much time do you spend playing sport?

Is this how much time spent per day, per week…?

Not all options covered by the tick boxes

1-5 times ☐ *6-10 times* ☐ *10 or more times* ☐

What if your answer is 0 hours?

Irrelevant questions

Q1. What is your address?

Do you need to know the persons address?

Biased questionnaires

Martha wants to test the following hypothesis.

'*More men than women buy a daily newspaper.*'

She plans to

- hand out a short questionnaire at a Women's Institute meeting,

Martha wants to compare men buying newspapers against women buying newspapers but is only giving the questionnaire to women.

Example: A survey is to be carried out to find the popularity of buying books with various age groups of the general population.
The survey is carried out by asking people question as they come out of a book shop.
Two questions from the survey questionnaire are shown below.

1. How old are you?
   Put a tick in the box.
   under 20 ☐
   20 to 30 ☐
   30 to 40 ☐
   older than 40 ☐

2. Do you buy books?
   Put a tick in the box.
   Yes ☐
   No ☐

a) Explain why this may be a biased survey.
The survey is being carried out outside a book shop therefore the answer to question 2 is more likely to be yes.

b) State a criticism about the design of question 1 in the survey.
The groups *under 20* and *older than 40* are too large. The intervals should be more spaced out.

Mathematics

Higher

Unit 39

Sampling

- Information is normally taken from a small part of a population. This is called a sample of the population.
- It is important to choose the sample without bias so that results will represent the whole population.
- It is cheaper and quicker to take samples, than to collect information from the whole population.
- The size of the sample is important. It needs to be large enough to represent the population but small enough to be manageable

Random Sampling

For a random sample, every member of the population has an **equal chance of being selected.**

Possible methods include picking names out of a hat (like a raffle) or using random numbers (from a table, calculator, or random number generator).

Example: The following list of random numbers was produced by using the random number button (RND) on a calculator. (All the digits were equally likely to be selected and were independent of each other.)

139 508 680 812 562 240 442 389 210 964 670 373 797 488 055

Use these numbers to randomly select a sample of 5 people out of 80.

- Firstly, give all the 80 people a 2-digit number from 01 to 80.
- Read the random digits in pairs to produce 2-digit numbers (13, 95, 08, 68, ...).
- Select those numbers in the range 01 to 80, reject those outside of the range and any repeated numbers.
- Stop when 5 have been selected.

13 ✓ 95 x 08 ✓ 68✓ 08 x 12✓ 56✓

The 5 selected people are those numbered 13, 08, 68, 12, 56.

This is what a good answer should look like! You MUST explain your steps!

Mathematics

Higher

Unit 39

Stratified Sampling

- A population may contain separate groups called strata. For stratified sampling, the population is divided into groups which have something in common e.g. school year groups.
- Each group needs to be fairly represented in the sample. The number from each group is proportional to the group size.
- The selection is then made at random from each group. A sample produced in this way is called a stratified sample.

To find out how many people will be chosen to represent each group in a sample, we use the formula: $\frac{\text{number in each group}}{\text{total population}} \times \text{sample size}$

You may have to round some of the values up or down in order to get the correct number in the sample.

Example: Bethan needs to survey 50 pupils from her school in order to gather opinions on school uniform. The numbers in each year group are given in the table.

Sample size = 50

Year group	7	8	9	10	11
Number of pupils	242	209	203	178	160

Calculate the number of pupils she should select from each year group.

Method:

Step 1: First, we work out the total

$242 + 209 + 203 + 178 + 160 = 992$

Step 2: Use the formula $\frac{\text{number in each group}}{\text{total population}} \times \text{sample size}$

Step 3: We round the numbers to the nearest whole number and check they add up to 50.

$12 + 11 + 10 + 9 + 8 = 50$

If they do NOT add up to your total, you will have to round one of the numbers differently to make sure they do.

Number from Year 7 $= \frac{242}{992} \times 50 = 12{\cdot}20$ Answer = 12 Year 7 pupils

Number from Year 8 $= \frac{209}{992} \times 50 = 10{\cdot}53$ Answer = 11 Year 8 pupils

Number from Year 9 $= \frac{203}{992} \times 50 = 10{\cdot}23$ Answer = 10 Year 9 pupils

Number from Year 10 $= \frac{178}{992} \times 50 = 8{\cdot}97$ Answer = 9 Year 10 pupils

Number from Year 11 $= \frac{160}{992} \times 50 = 8{\cdot}06$ Answer = 8 Year 11 pupils

Mathematics

Higher

Unit 39

Systematic Sampling

- Systematic sampling means taking one item from a list at regular fixed intervals e.g. every 5th, every 20th, etc.
- It is useful in certain situations e.g. in regularly testing the quality of items manufactured in a factory.
- It is important to understand that systematic sampling is NOT random as all items are not equally likely to be selected

e.g. if it is decided to start with the 1st item and then select every 10th, this means the 2nd, 3rd, ... cannot be selected.

Example: Explain how you would pick a systematic sample of 10 items from a total of 120 items.

Answer:

- Number each item 1 – 120
- Divide total number by the number in the sample: 120 ÷ 10 = 12
- Pick the first item at random (easier to pick one from the first 12)
- Pick every 12^{th} item after the first one

Mathematics

Higher

Unit 40

Scatter Diagrams

A **Scatter Diagram** shows the relationship between two variables. **Correlation** is used to describe the relationships.

Remember: When choosing a scale, make sure you always go up in **equal steps** along each axis.

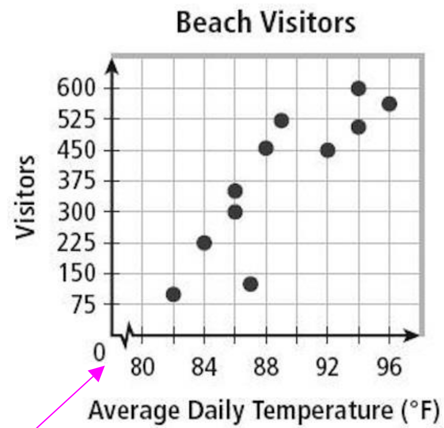
Drawing a Scatter Diagram

Method

- Decide on the scale you are going to use for the 1st set of data. This is usually on the horizontal axis.
- Decide on the scale you are going to use for the 2nd set of data. This is usually on the vertical axis.

  (Note: It does not really matter which set of data goes on the x axis and which on the y;
  I would recommend putting the one with the biggest numbers on the y axis.
  Remember to label both axes, including units.

- The vertical axis does not have to have the same scale as the horizontal axis, but each axis must have a 'uniform scale'.
- Each axis does not need to start from zero.
- The values are placed on the lines not in the spaces.
- Complete both axes and do not forget to LABEL fully.
- Plot the points carefully and mark with a dot or cross. Do not join up the points.



This symbol means a chunk has been taken out of the axis – which means it does not have to start at zero.

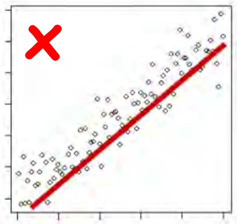
The Line of Best Fit

This is a single straight line which is supposed to be a good representation of the pattern / trend of the data.

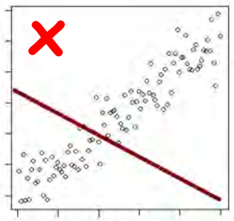
When drawing the line of best fit:

- Make sure the line follows the trend of data.
- Try to get roughly the same amount of points above the line as below
- Experiment by using your ruler as your line, and only draw the line in when you are happy
- Do not spend too long deciding, and do not try to make it perfect.

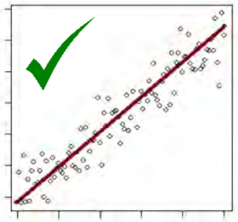
Note: Your line does NOT have to start at the origin (0, 0)



A lot more points above than below



Does not follow the trend of data



Good line of best fit

Mathematics

Higher

Unit 40

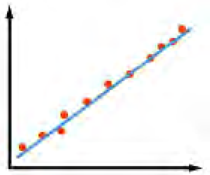
Correlation

The most important use of scatter diagrams is to determine the type (if any) of correlation between two variables

Correlation is the relationship between the two variables.

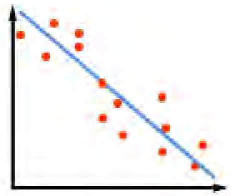
Positive Correlation

As one variable increases, so does the other.



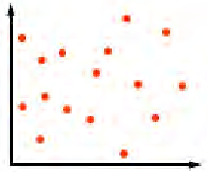
Negative Correlation

As one variable increases, the other decreases.



No Correlation

No relationship between the variables.



It is also worth noting the strength of the correlation.

STRENGTH

Strong – dots are close to each other

Weak – dots are far apart

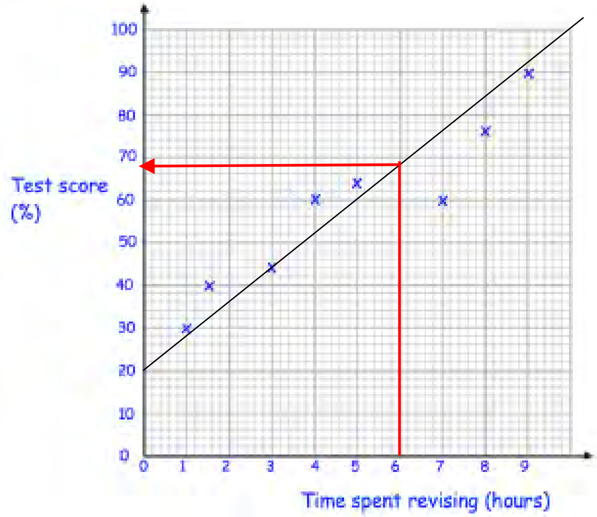
We can use the line of best fit and the correlation to predict results we don't already have.

Note: The stronger the correlation, the more reliable these predictions will be.

Example 1:

Below is a table showing the time each pupil spent revising and the test score they achieved.
Draw a scatter diagram and include the line of best fit.

Time (hours)	1.5	4	8	1	5	9	7	3
Test Score (%)	40	60	76	30	64	90	60	44



a) What type of correlation is shown?

Positive Correlation

b) Another student spent 6 hours revising for the test. Find an estimate of their test score.

Draw a line of best fit and read from it - 68%

c) Explain why it might not be sensible to use the scatter graph to estimate the score for a student that spent 15 hours revising.

It is out of the data range.

Mathematics

Higher

Unit 40

Sometimes you may be asked to draw the line of best fit through the mean point. Remember to find the mean, add up all the values and divide by how many there are.

Plot the mean of one variable against the mean of the other variable and ensure your line of best fit goes through this coordinate.

Example 2:
Below is a table showing the temperature on certain summer days and the number of ice creams sold on those particular days.
Draw a scatter diagram for the given information.

Temp (℃)	28	25	26	21	23	29	27	29
Ice creams sold	27	22	25	10	14	33	23	30

a) What type of correlation is shown?

Positive Correlation

b) The mean point for the temperature is 26℃, calculate the mean number of ice creams sold.

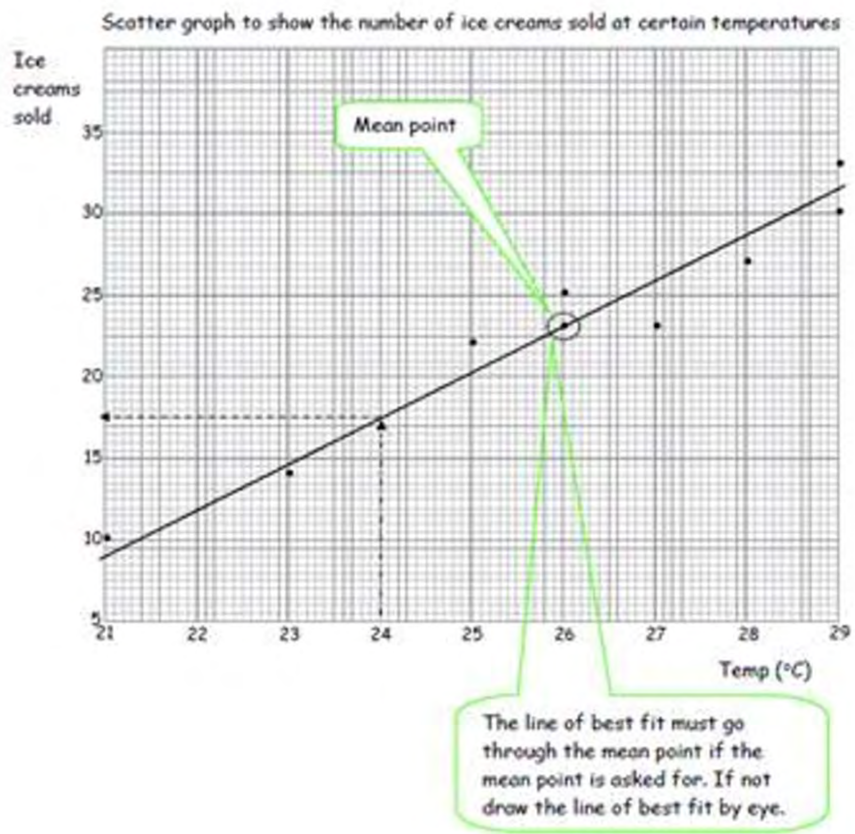
Mean = (27 + 22 + 25 + 10 + 14 + 33 + 23 + 30) ÷ 8
= 184 ÷ 8
= 23

c) Draw the line of best fit

Plot the mean points against one another first and the line of best fit must go through this.

d) If the temperature was 24℃, estimate the number of ice creams sold.

Read from 24℃ up to the line of best fit and across.
There were 17 ice creams sold



Mathematics

Higher

Unit 41

Sequences

A **sequence** is a set of numbers that follow **a pattern or a rule**.
Each number in a sequence is called a **term**.

A sequence with a **common difference** (increases or decreases by the same amount each time) is called a **linear sequence (e.g. 8, 11, 14, 17…)**.
If it does not change by the same amount it is a **non-linear sequence** (e.g. 5, 15, 45, 135…).

There are certain sequences you should recognise such as:

Square numbers – 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144…   and Cube numbers – 1, 8, 27, 64, 125…

Recognising and Continuing Sequences

You will be given a sequence of numbers. You need to look at the difference between each term of the sequence to work out its rule.

Use the same rule to get the next term. If there is a common difference, look at adding or subtracting. If the difference between terms changes, try multiplying or dividing.

Term-to-Term Rule

The rule in example 1 shows how we get from one term to the next.
Therefore, the **term-to-term rule** of the sequence 4, 7, 10, 13, 16, … is +3

Example 1: Find the next three terms of the sequence:

4, 7, 10, 13, 16…

Find the difference between each term

4, 7, 10, 13, 16…

In this case we add 3

Calculate the next three terms

4, 7, 10, 13, 16, 19, 22, 25

Mathematics

Higher

Unit 41

Position-to-term rule

Each term in a sequence has a position. The first term is in position 1, the second term is in position 2 and so on.

Each term is linked to the position in which it lies in the sequence. The position-to-term rule describes that link. Your rule must work for every term.

Example 2: Find the position-to-term rule for 4,5,6,7,8….

Note the position of each term

Position	1	2	3	4	5
Sequence (Terms)	4	5	6	7	8

Work out how to go from the position to the term. Remember it must work for every term.

The position-to-term rule is +3.

The n^{th} term

The n^{th} term of a sequence is the position-to-term rule using n to represent the position number. You may be asked to generate a sequence from an n^{th} term and find the n^{th} term of a given sequence.

Notice in example 3 that the linear sequence goes up in 3's and the n^{th} term was $3n$ – 2.

Generating a Sequence from the n^{th} Term

Example 3: The n^{th} term for a sequence is $3n$ – 2. What are the first 5 terms of the sequence?

Substitute n for each position number

Position	1	2	3	4	5
Workings out	3 × 1 - 2	3 × 2 - 2	3 × 3 - 2	3 × 4 - 2	3 × 5 - 2
Sequence	1	4	7	10	13

For the 1st term swap n for 1

For the 4th term swap n for 4

The sequence is 1, 4, 7, 10, 13…

Mathematics

Higher

Unit 41

Notice in example 4 the sequence is non-linear (the difference between terms are different). That is because it was a **Quadratic n^{th} term**, the n was squared.

Now we will look at finding the n^{th} term. There are two methods.

Generating a Sequence from the n^{th} Term

Example 4: The n^{th} term for a sequence is $2n^2$ - 3.

What are the first 5 terms of the sequence?

Position	1	2	3	4	5
Workings	2×(1)²-3	2×(2)²-3	2×(3)²-3	2×(4)²-3	2×(5)²-3
Sequence	-1	5	15	29	47

Apply BIDMAS. As the square is only attached to the n, square the n first then multiply by 2!

The sequence is -1, 5, 15, 29, 47…

Finding the n^{th} term (linear sequence)

Example 5a: Find the n^{th} term of this sequence.

5, 7, 9, 11, 13

Option 1:

Note the difference between each term

5, 7, 9, 11, 13

+2 +2 +2 +2

This number goes in front of the n $2n$

Subtract your number from the first sequence number 5 – 2 = 3
This is the second part of your nth term

The n^{th} term is $2n + 3$

Finding the n^{th} term (linear sequence)

Example 5b: Find the n^{th} term of this sequence.

5, 7, 9, 11, 13

Option 2:

Note the difference between each term

5, 7, 9, 11, 13

+2 +2 +2 +2

This number goes in front of the n $2n$

Substitute in 1, work out how to get from your number to the first term in the sequence. 2 x 1 = 2 We need 5 so we add 3

The n^{th} term is 2n + 3

Mathematics

Higher

Unit 41

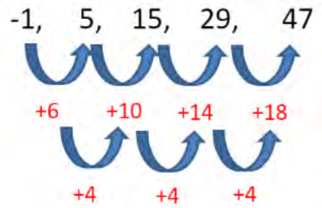
In example 5, the sequence had a common difference.

If the difference between terms changes then it has a quadratic n^{th} term, like example 6.

When you have found the nth term, substitute in some values to check.

Finding the n^{th} term (non-linear/quadratic sequence)

Example 6: Find the nth term of this sequence. **-1, 5, 15, 29, 47**



If you notice the term difference is not constant find the second difference.

Because a second difference is needed, we call the sequence quadratic, and we know the equation will have n^2 in it.

Half the second difference. 4 ÷ 2 = 2   The sequence begins with $2n^2$

As before, subtract the number in front of n from the first term of the sequence (or use the other method). -1 – 2 = -3

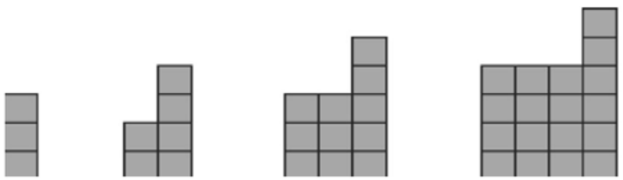
The n^{th} term is $2n^2$ - 3

You may also be asked to find a specific number, for example what is the 50th term in the sequence $4n - 6$.

Substitute in $n = 50$ $4 \times 50 - 6 = 194$

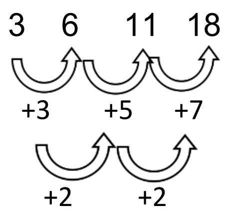
You may also need to apply this knowledge to diagrams, as shown in example 7.

Example 7: How many squares would be in the next diagram?



Convert your diagrams into a numerical sequence
3, 6, 11, 18

Find the n^{th} term



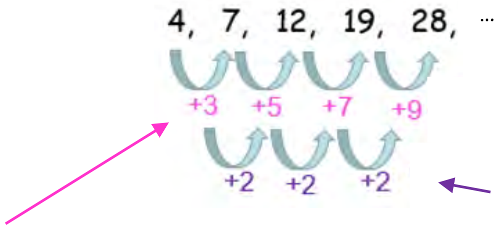
n^{th} term is $n^2 + 2$

Find how many squares are in the 5th diagram

$5^2 + 2 = 27$ squares

Of course, you may have got the nth term by recognising the sequence of square numbers with 2 added squares on top!

Example 8: Find the nth term of this sequence. **4, 7, 12, 19, 28, …**



The second difference

The sequence does not go up or down in nice equal steps, the steps get bigger each time. Therefore, we find the second difference.

If a second difference is needed , this tells us that the nth term is quadratic (there is an n^2 in it).

Half the second difference. This becomes the coefficient of n^2 (number in front of the n^2).

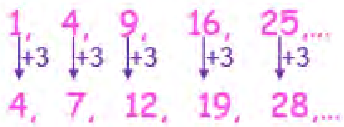
Half of 2 is 1, but there is no need to write the 1 in front. n^2

Substitute $n = 1, n = 2, n = 3, n = 4, \ldots$ into the equation so far:

$1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 16, \quad 5^2 = 25$

This gives us the sequence: 1, 4, 9, 16, 25, …

Look at what you need to do to this sequence to make the sequence in the question.



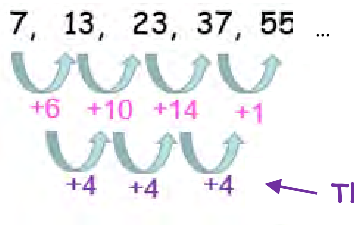
So, the n^{th} term is: $n^2 + 3$

Mathematics

Higher

Unit 41

Example 9: Find the nth term of this sequence. **7, 13, 23, 37, 55, …**



The second difference

The second difference is needed , this tells us that the nth term is quadratic (there is an n^2 in it, $2n^2$ in this case).

Half the second difference. This becomes the coefficient of n^2 (number in front of the n^2).

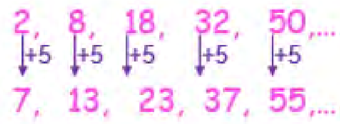
Half of 4 is 2, so put 2 in front of n^2. $2n^2$

Substitute $n = 1, n = 2, n = 3, n = 4, \ldots$ into the equation so far

$$2 \times 1^2 = 2, \quad 2 \times 2^2 = 8, \quad 2 \times 3^2 = 18, \quad 2 \times 4^2 = 32, \quad 2 \times 5^2 = 50$$

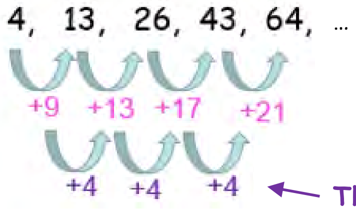
This gives us the sequence: 2, 8, 18, 32, 50, …

Look at what you need to do to this sequence to make the sequence in the question.



So, the n^{th} term is: $2n^2 + 5$

Example 10: Find the nth term of this sequence. **4, 13, 26, 43, 64, …**



The second difference

Half the second difference. This becomes the coefficient of n^2 (number in front of the n^2).

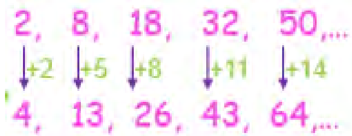
Half of 4 is 2, so put 2 in front of n^2. $2n^2$

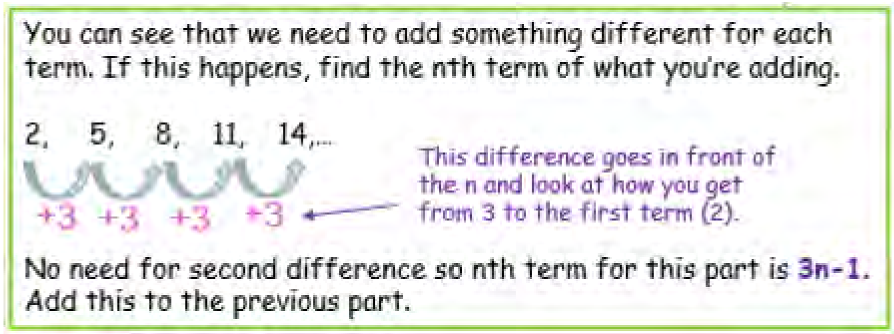
Substitute $n = 1, n = 2, n = 3, n = 4, \ldots$ into the equation so far

$2 \times 1^2 = 2, \quad 2 \times 2^2 = 8, \quad 2 \times 3^2 = 18, \quad 2 \times 4^2 = 32, \quad 2 \times 5^2 = 50$

This gives us the sequence: 2, 8, 18, 32, 50, …

Look at what you need to do to this sequence to make the sequence in the question.




You can see that we need to add something different for each term. If this happens, find the nth term of what you're adding.

2, 5, 8, 11, 14,…

+3 +3 +3 +3

This difference goes in front of the n and look at how you get from 3 to the first term (2).

No need for second difference so nth term for this part is **3n-1**. Add this to the previous part.

So, the n^{th} term is: $2n^2 + 3n - 1$

Mathematics

Higher

Unit 43

Probability

Probability is the likelihood that an event will occur.

Probabilities are always written as fractions, decimals, or percentages.

Probabilities have values between 0 and 1.

Probability scale

Probabilities can be described using words

Impossible — Unlikely — Even Chance — Likely — Certain

What is the probability that if you toss a coin it will land on heads?
Even Chance (the coin is equally likely to land on heads or tails)

What is the probability that Winter follows Summer?
Impossible

What is the probability that Christmas will be on the 25th of December?
Certain

Probabilities can also be described using numbers

Impossible	Unlikely	Even	Likely	Certain
0		1/2		1
0		0.5		1.0
0%		50%		100%

The probability of an event happening can be found using:

$$\text{P(event happening)} = \frac{\text{number of ways the event could happen}}{\text{the total number of outcomes}}$$

Example 1: Find the probability of throwing an even number on a dice.

$\text{P(even number)} = \frac{3}{6}$

3 ← Number of even numbers on a dice (2, 4, 6)

6 ← Total amount of numbers on a dice

Example 2: What is the probability of picking a diamond from a full deck of cards?

$\text{P(diamond)} = \frac{13}{52}$

13 ← Number of diamonds in a pack of cards

52 ← Total number of cards in a pack of cards

The probability of an event **not** happening can be found using:

P(event not happening) = 1 – P(event happening)

Example: What is the probability of **not** picking a diamond from a full deck of cards?

P(not diamond) = 1 – P(diamond)

$= 1 - \frac{13}{52} = \frac{39}{52}$

Mathematics

Higher

Unit 43

Listing Outcomes

You might be asked to list all the possible outcomes for two or more events.

Example: List all the 3-digit numbers that can be made using the digits 3, 6, and 9?

369 396 639 693 936 963

Example: A coin is flipped, and a dice is rolled. List all the possible outcomes.

A head on the coin → H 1

A 1 on the dice → T 1

H 1 H 2 H 3 H 4 H 5 H 6

T 1 T 2 T 3 T 4 T 5 T 6

A tail on the coin → H 6

A 6 on the dice → T 6

Sample Space Diagram

A sample space diagram is a way of showing multiple outcomes in one diagram.

Example: Two dice are thrown, and the numbers are multiplied together. The table below shows some of the possible outcomes.

Second Dice						
6	6	12	18	24	30	36
5	5	10	15	20	25	30
4	4	8	12	16	20	24
3	3	6	9	12	15	18
2	2	4	6	8	10	12
1	1	2	3	4	5	6
	1	2	3	4	5	6

First Dice

First dice x second dice 5 x 6 = 30

First dice x second dice 6 x 3 = 18

a) Complete the table to show **all** the possible outcomes.

b) What is the probability of getting an outcome that is an odd number?

$P(\text{odd}) = \frac{9}{36} = \frac{1}{4}$

Number of outcomes that are odd numbers

Total number of outcomes

c) If the two dice were thrown a total of 60 times, how many times would you expect to get an outcome greater than 10?

$P(6 \text{ and } H) = 60 \times \frac{9}{36} = 15$ times

Number of times the dice are thrown

Probability of an odd number

Finding Missing Probabilities from a Table

Probabilities add up to 1, to find the missing probabilities add together the probabilities you are given and subtract them from 1.

Example: A biased spinner has 4 colours. The probability of the spinner landing on each colour is given below.

Colour	Red	Blue	Yellow	Green
Number of times	0.1	x	0.4	0.2

a) What is the probability of choosing a blue sweet?

P(Blue) = 0.1 + 0.4 + 0.2 = 0.7 ← Add the probabilities

1 – 0.7 = 0.3 ← Subtract them from 1

b) The spinner is spun 100 times. Calculate an estimate for the number of times the spinner will land on yellow.

P(Yellow) = 100 x 0.4 = 40 times

Number of times the spinner is spun

Probability of a yellow

Mathematics

Higher

Unit 43

The **'AND'** / **'OR'** rules

Key words

Independent events – one event happening does not change the probability of the other one happening

Mutually exclusive events – events that are not able to happen at the same time as each other

The '**AND**' rule

If you want one outcome **and** the other outcome, then you multiply their probabilities

For two independent events A and B

P(A **and** B) = P(A) **x** P(B)

Example:

Find the probability of rolling a 6 on a dice **and** getting a head on the toss of a coin

P(6 **and** H) = $\frac{1}{6} \times \frac{1}{2}$

$= \frac{1}{12}$

The '**OR**' rule

If you want one outcome **or** the other outcome, then you add their probabilities

For two mutually exclusive events A and B

P(A **or** B) = P(A) **+** P(B)

Example:

The table below shows the probability that a spinner lands on a certain colour

Colour	Yellow	White	Blue	Red
Probability	0.2	0.25	0.15	0.4

What is the probability that the spinner lands on yellow or red?

P(Y **or** R) = 0.2 + 0.4

= 0.6

Relative Frequency

Some probabilities can be estimated by doing experiments or trials, this is called relative frequency.

The more trials that are done (100+), the more accurate the estimated probability will be.

$$\text{Relative Frequency} = \frac{\text{number of times the event occurs}}{\text{total number of trials}}$$

Relative frequency from a table

Example:

A spinner is spun 100 times. The colour on the spinner is recorded after each spin. The table below shows the results recorded.

Colour	White	Green	Blue
Frequency	21	52	27

What is the relative frequency of spinning a green?

$\frac{52}{100} = 0.52$

Total number of spins

Mathematics

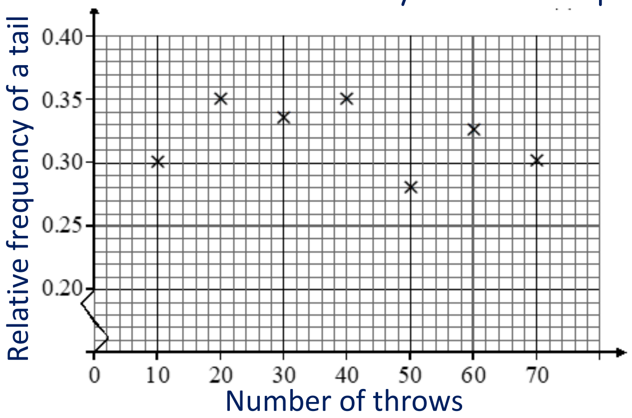
Higher

Unit 43

Relative frequency from a graph

Example:

A coin is thrown 70 times. The relative frequency of the number of tails after every 10 throws is plotted.



a) How many tails were obtained in 50 throws?

0.28 x 50 = 14 tails

Relative frequency of 50

Number of throws

b) Use the diagram to estimate the probability of getting a tail.

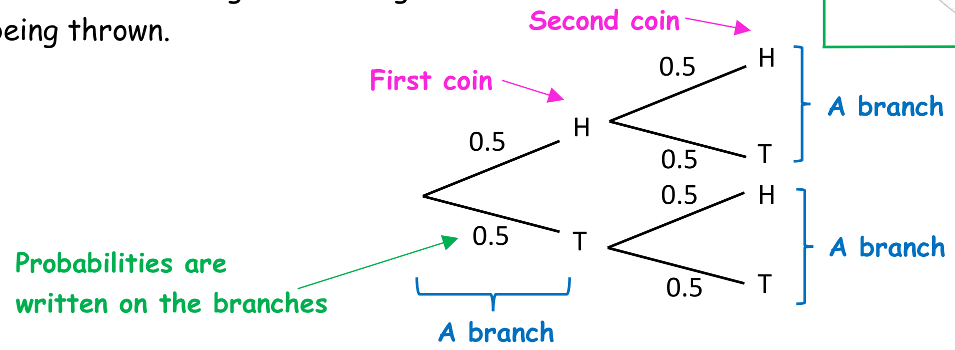
P(T) = 0.3 (the more throws the more accurate the result, 70 throws = 0.3)

Probability Trees

Tree diagrams are a way of showing combinations of two or more events.

Tree diagrams have branches, with each branch adding to 1.

Here is a tree diagram showing the outcomes of two coins being thrown.



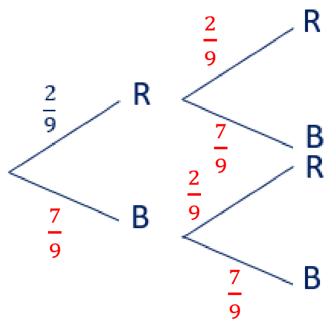
Example:

A bag contains red and blue counters. The probability that a red counter is chosen is $\frac{2}{9}$.

A counter is chosen and replaced; a second counter is chosen.

a) Complete the tree diagram below.

remember each branch adds to 1



b) Calculate the probability that a red counter is chosen followed by a blue counter.

We need to follow the branches, red for the first counter **AND** blue for the second counter.

P(R **and** B) $= \frac{2}{9} \times \frac{7}{9} = \frac{14}{81}$

c) Calculate the probability that two counters of the same colour are chosen.

P(R **and** R) $= \frac{2}{9} \times \frac{2}{9} = \frac{4}{81}$

OR

P(B **and** B) $= \frac{7}{9} \times \frac{7}{9} = \frac{49}{81}$

$\frac{4}{81} + \frac{49}{81} = \frac{53}{81}$

Mathematics

Higher

Unit 43

Conditional Probability

Conditional Probability

Conditional probability is when the probability of an event occurring depends on the outcome of another event.

For example, the probability that I take an umbrella to work could depend on the probability of it raining on a given day.

Example 1: A bag contains 4 red balls and 1 blue ball. A ball is removed from the bag at random and its colour is noted. The ball is NOT replaced in the bag. Another ball is chosen at random.

What is the probability that the second ball selected is red?

This is conditional probability because the chance of getting a red ball the second time depends on which ball was chosen the first time. If a red ball has already been removed there are only 3 red balls left whereas if a blue ball was chosen the first time, then all of the balls left are red. We need to work out both possible scenarios.

P(blue then red) = P(blue first) x P(red second)

$= \frac{1}{5} \times \frac{4}{4}$ ← Once 1 blue has been chosen, there are only 4 balls left in the bag.

$= \frac{1}{5} \times 1$

$= \frac{1}{5}$

P(red then red) = P(red first) x P(red second)

$= \frac{4}{5} \times \frac{3}{4}$ ← Once 1 red has been chosen, there are only 3 red balls left and only balls left in the bag.

$= \frac{12}{20}$

$= \frac{3}{5}$

P(second is red) $= \frac{1}{5} + \frac{3}{5} = \frac{4}{5}$

Example 2: The probability that Tom catches the bus to school is 0.7. If Tom catches the bus to school, the probability that he is is late for school is 0.2. If Tom does not catch the bus, the probability that he's late is 0.4.

What is the probability that Tom is late for school on any given day?

Again, we need to work out both possible scenarios.

P(catches bus and late) = P(catches bus) x P(late)

= 0.7 x 0.2

= 0.14

P(catches bus and late) = P(doesn't catch bus) x P(late)

= 0.3 x 0.4

= 0.12 ← P(doesn't catch bus) = 1 – 0.7

P(late) = 0.14 + 0.12 = 0.26

Mathematics

Higher

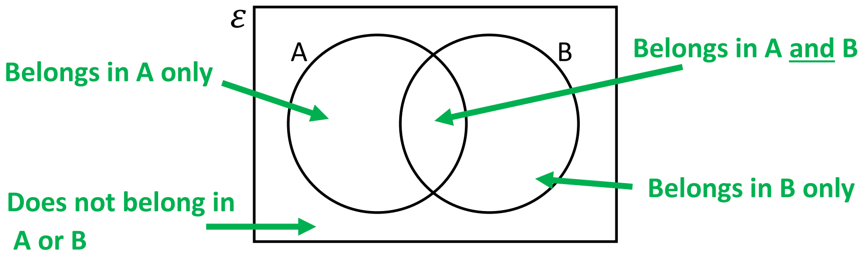
Unit 43

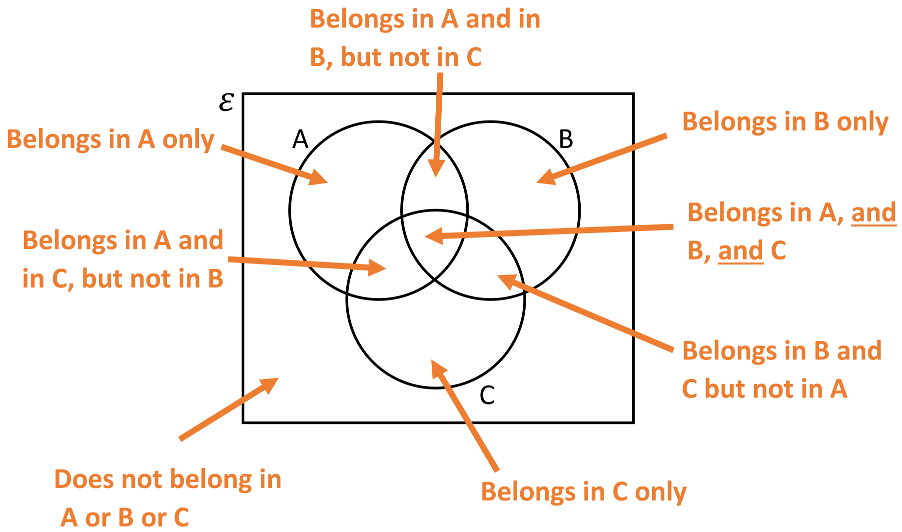
Venn Diagrams

A Venn diagram provides a means of classifying items of data which may or may not share common properties.
The universal set, ε, contains everything we are interested in at that time – it contains all the data we need to use for each individual question.

Drawing Venn Diagrams

A Venn diagram consists of 2 or 3 overlapping circles surrounded by a box.





Example: Display the following information in a Venn diagram.
Universal Set (ε): Integers between 5 and 15 inclusive
Set A: Multiples of 5
Set B: Odd numbers

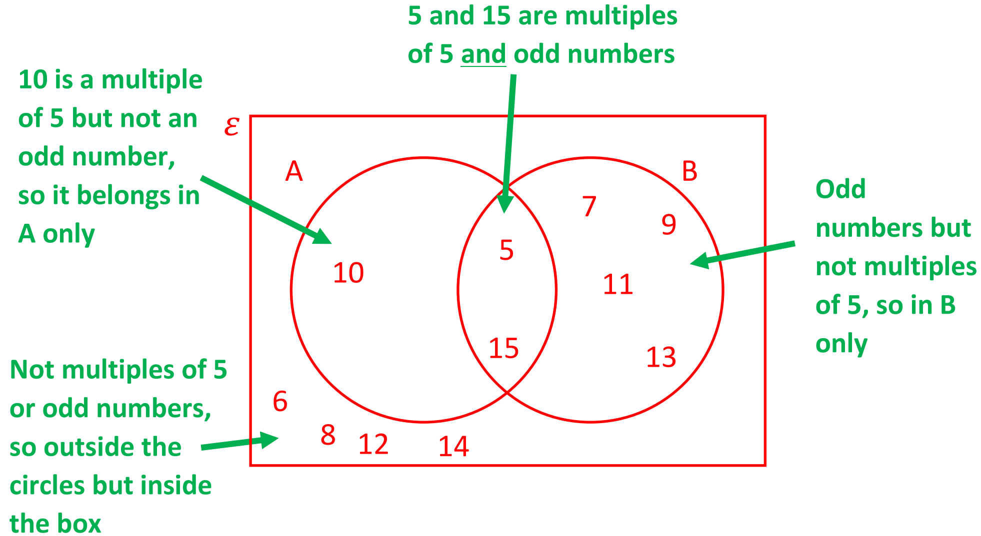
Whole numbers

All the numbers from 5 to 15 including the 5 and the 15

Step 1: Draw the Venn diagram, label the circles.

Step 2: From the numbers given:

- Write any that are multiples of 5 AND odd numbers in the centre area.
- Write any other multiples of 5 in the circle representing set A.
- Write any other odd numbers in the circle representing set B.
- Any numbers not yet used go outside the circles but inside the box.



Mathematics

Higher

Unit 43

Venn Diagrams – Two Circles, Given the Number in the Overlapping Section

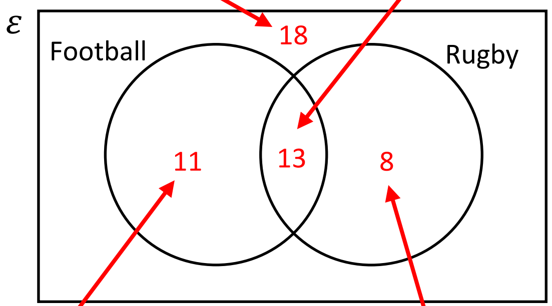
Example 1: In a group of 50 students:

- 24 play football
- 13 play both football and rugby
- 18 play neither football nor rugby

Complete the Venn diagram

Step 1: Fill in the number that goes outside the circles but inside the box.

Step 2: Fill in the centre section of the circles – the "both" section



Step 3: The whole of the football circle represents 24 students. There are already 13 of them in the overlapping area.
24 – 13 = 11.
There are 11 football students left.

Step 4: There are 50 students altogether.
18 + 13 + 11 = 42.
We have already used 42 students. 50 – 42 = 8.
There are 8 rugby students left.

Example 2: In a group of 140 students:
90 say Mathematics is their favourite subject

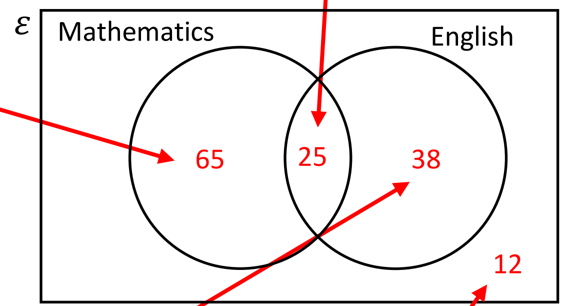
63 say English is their favourite subject

25 say both Mathematics and English are their favourite subjects

Complete the Venn diagram

Step 1: Fill in the centre section of the circles – the "both" section

Step 2: The whole of the Mathematics circle represents 90 students. There are already 25 of them in the overlapping area.
90 – 25 = 65. There are 65 Mathematics students left.



Step 3: The whole of the English circle represents 63 students. There are already 25 of them in the overlapping area.
63 – 25 = 38.
There are 38 English students left.

Step 4: There are 140 students altogether.
65 + 25 + 38 = 128.
We have already used 128 students.
140 – 128 = 12.
There are 12 students left.

Mathematics

Higher

Unit 43

Venn Diagrams – Two Circles, Not Given the Number in the Overlapping Section

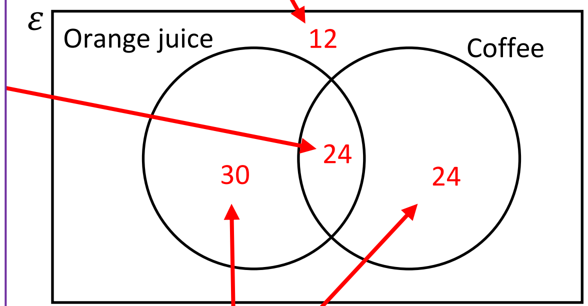
Example 3: There were 90 people at a breakfast buffet

- 54 drank orange juice
- 48 drank coffee
- 12 did not drink either orange juice or coffee

Complete the Venn diagram

Step 1: Fill in the number that goes outside the circles but inside the box.

Step 2: We have already used 12 people.
90 – 12 = 78.
There are 78 people left to use.
If we add the number of people who drank orange juice and the number of people who drank coffee, we get
54 + 48 = 102.
This is more than the 78 we have left to use.
Find the difference
102 – 78 = 24
This difference goes inside the overlapping section – the "both" section



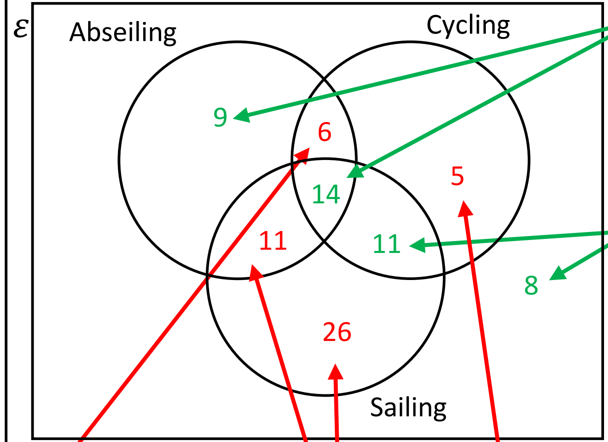
Step 3 and 4: The whole of the orange juice circle represents 54 people.
There are already 24 of them in the overlapping area.
54 – 24 = 30.
There are 30 orange juice people left.

The same with the Coffee people.
48 – 24 = 24.

Venn Diagrams – Three Circles

Example: 90 teenagers took part – or not – in various activities.
40 teenagers did abseiling
14 did all three activities and 8 did none of them
36 did cycling
9 did only abseiling
20 did abseiling and cycling
11 only did sailing and cycling

Step 1: We can fill these in straight away. 14 did all 3. 8 did none. 9 did **only** abseiling, meaning abseiling but not cycling and not sailing.
11 **only** did sailing and cycling, this does not include abseiling – it does not include the centre overlapping section



Step 3: Look at each circle in turn, always start with the circle which has just one section left to fill in

Step 2: We need to see if we can fill any other overlapping sections in first before we look at the circles as a whole.
20 did abseiling and cycling, this does include the centre overlapping section.
There are 14 teenagers already in the centre.
20 – 14 = 6

The cycling circle has one section left to fill in. 35 did cycling. This is the whole cycling circle.
6 + 14 + 11 = 31
We have already got 30 teenagers in the cycling circle.
36 – 31 = 5.

The abseiling circle has one section left to fill in. 40 did abseiling. This is the whole abseiling circle.
9 + 6 + 14 = 29
We have already got 29 teenagers in the abseiling circle.
40 – 29 = 11.

And now the sailing circle only has one section left to fill in.
We need to see how many teenagers are left.
9 + 6 + 5 + 11 + 14 + 11 + 8 = 64
90 – 64 = 26.

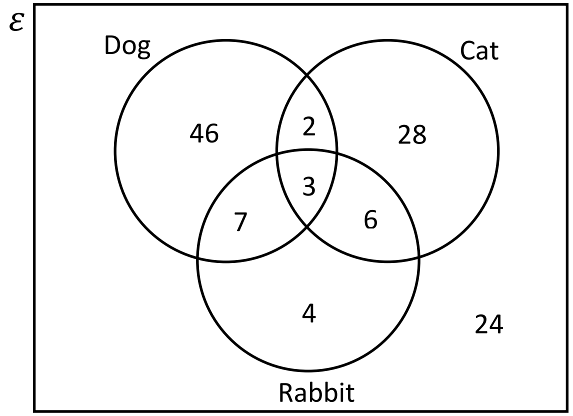
Mathematics

Higher

Unit 43

Probability and Interpreting Venn Diagrams

Example: A group of people were asked what pet they had. The answers are shown in the Venn diagram below. The universal set, ε, contains all the people in the group.



a) How many of the people had exactly one pet?

Exactly one pet means only a dog (46) or only a cat (28) or only a rabbit (4).

P(D **or** C **or** R) = 46 + 28 + 4 = 78

b) How many people had a cat and a dog?

A cat and a dog is the overlapping section between the dog circle and the cat circle, as the question does not say **only** a cat and a dog it includes the middle overlapping section as well, where all three circles overlap. P(C and D) = 2 + 3 = 5

c) One of the people is chosen at random. What is the probability that the person has a dog? The question does not say **only** a dog, so we are looking at the whole dog circle. 46 + 2 + 3 + 7 = 58.

The total number of people asked is

46 + 2 + 3 + 7 + 28 + 6 + 4 + 24 = 120

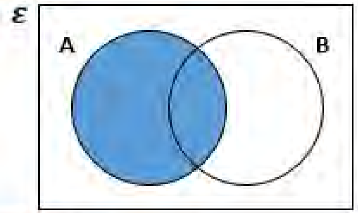
$$P(\text{dog}) = \frac{58}{120} \text{ or } \frac{29}{60}$$

d) One of the people is chosen at random. What is the probability that the person did not have a cat or a dog or a rabbit?

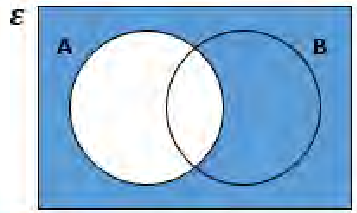
$$P(\text{no pet}) = \frac{24}{120} \text{ or } \frac{1}{5}$$

Set notation

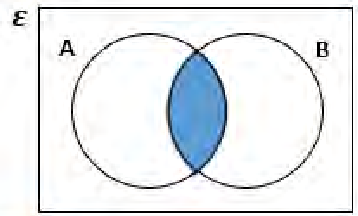
A — Everything in set A



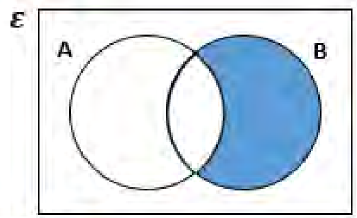
A′ — Everything not in set A (The complement of A)



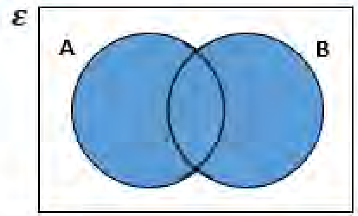
A ∩ B — A intersect B. Everything in both



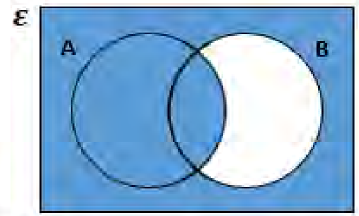
A′ ∩ B — Everything in B and not A



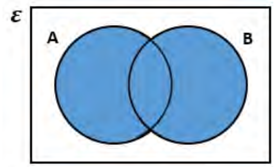
A ∪ B — A union B. Everything in A or B



A ∪ B′ — Everything in A or not in B



Example: Which of the following sets represents the **shaded** area in the Venn diagram below? Circle your answer



A $\quad B'$ $\quad$ (circled) $A \cup B$ $\quad A \cap B$ $\quad A' \cap B$

Mathematics

Higher

Unit 44

Histograms

Drawing Histograms

There are 5 differences between bar charts and histograms:

- Bar charts have gaps between bars, histograms have no gaps between bars
- The horizontal axis on a histogram has a continuous scale
- The **area** of each bar in a histogram represents the frequency
- The vertical axis on a histogram is labelled **frequency density**
- Histograms are used when some of the groups have **different widths**. It makes it easier to see which bar/group has the highest frequency.

To work out the frequency densities we use the formula:

$$\text{Frequency Density} = \frac{\text{Frequency}}{\text{Group Width}}$$

Frequency = area of a bar

Frequency = Group width x frequency density

To work out group width, we just subtract the lower limit of each group from the upper limit of the group.

We then plot the frequency density on the y-axis and our data is on the x-axis

Note: In histograms, **the bars MUST to touching**.

Mathematics

Higher

Unit 44

Example: Mr Burden has announced to Ysgol Cwm Brombil that there will be no homework over the summer holiday. The lengths of applause from different groups of pupils is shown in the table below. Draw a histogram to display these results.

Length of applause (mins)	Frequency
$0 < a \leq 1$	2
$1 < a \leq 2$	4
$2 < a \leq 3$	15
$3 < a \leq 5$	10
$5 < a \leq 8$	6
$8 < a \leq 13$	5

Add an extra column for Frequency Density.

$$\text{Frequency Density} = \frac{\text{Frequency}}{\text{Group Width}}$$

These numbers go on the x-axis (as a scale from 0 to 13, NOT as separate groups)

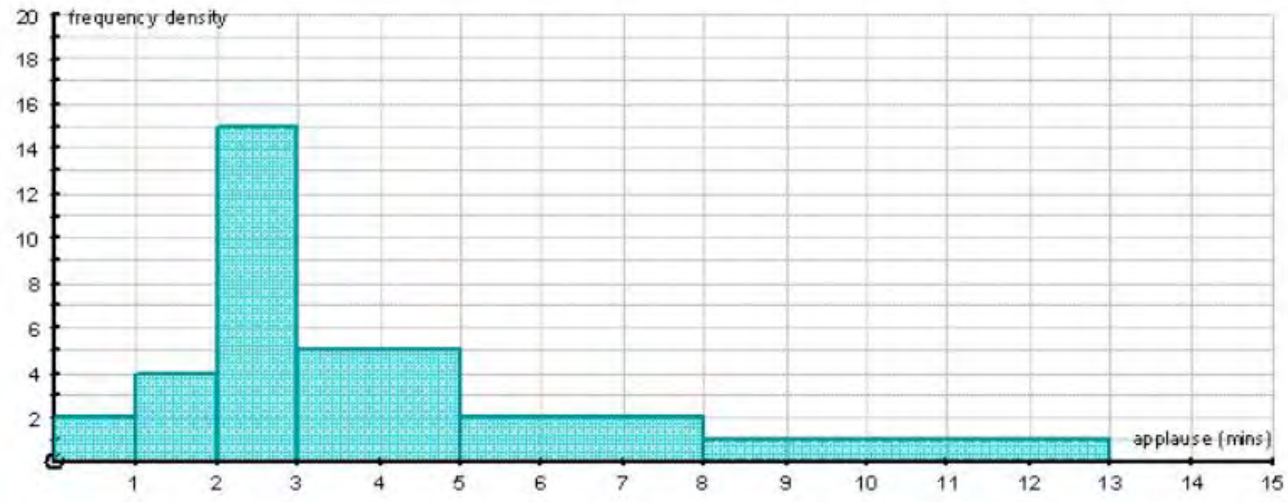
Plot these values on the y-axis

Group width: $1 - 0 = 1$

Group width: $13 - 8 = 5$

Length of applause (mins)	Frequency	Frequency Density
$0 < a \leq 1$	2	$2 \div 1 = 2$
$1 < a \leq 2$	4	$4 \div 1 = 4$
$2 < a \leq 3$	15	$15 \div 1 = 15$
$3 < a \leq 5$	10	$10 \div 2 = 5$
$5 < a \leq 8$	6	$6 \div 3 = 2$
$8 < a \leq 13$	5	$5 \div 5 = 1$

A Histogram to show the Length of Applause after Mr Burden says "No Homework"



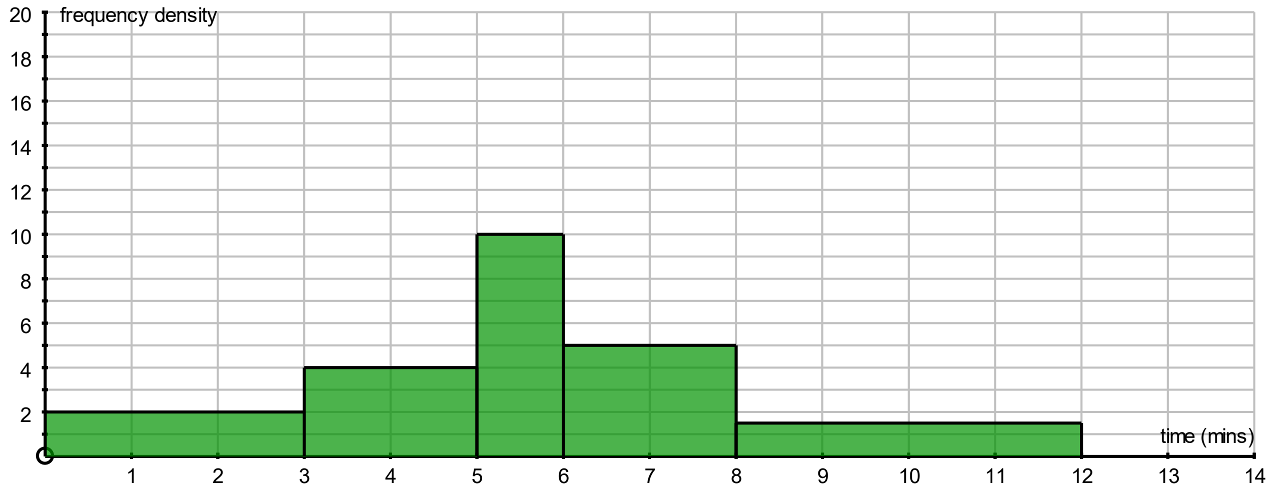
Mathematics

Higher

Unit 44

Interpreting Histograms

Example: Here is a Histogram showing the time taken by some year 7's to complete all their times tables. Find the frequencies of each group.



This time we are given Frequency Density and asked to work out Frequency. If we do a little re-arranging to our formula we get:

Frequency = Frequency Density × Group Width

Time (mins)	Frequency Density	Group Width	Working	Frequency
$0 < t \le 3$	2	3	$2 \times 3 = 6$	**6**
$3 < t \le 5$	4	2	$4 \times 2 = 8$	**8**
$5 < t \le 6$	10	1	$10 \times 1 = 10$	**10**
$6 < t \le 8$	5	2	$5 \times 2 = 10$	**10**
$8 < t \le 12$	1.5	4	$1.5 \times 4 = 6$	**6**

Mathematics

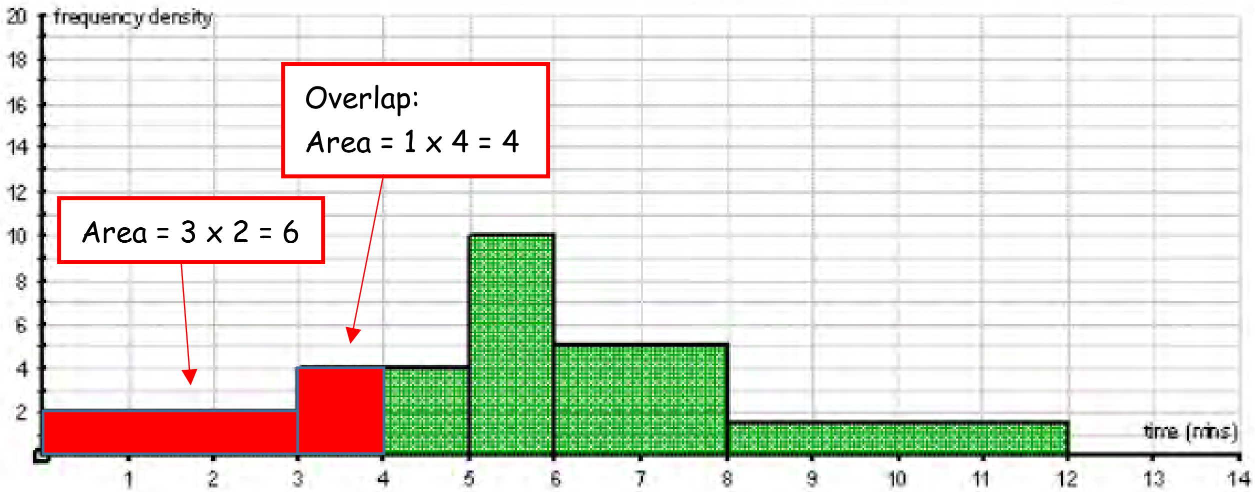
Higher

Unit 44

If the question asked us how many Year 7s completed their times tables in LESS THAN 6 mins, we just add up the frequencies for the first three bars. 6 + 8 + 10 = 24

If the question asked us how many Year 7s completed their times tables in MORE THAN 6 mins, we just add up the frequencies for the last two bars. 10 + 6 = 16

But what if the question asked us how many Year 7s completed their times tables in LESS THAN 4 mins? This is in the middle of bars! We need to work out the area of the bars up until that point! 6 + 4 = 10



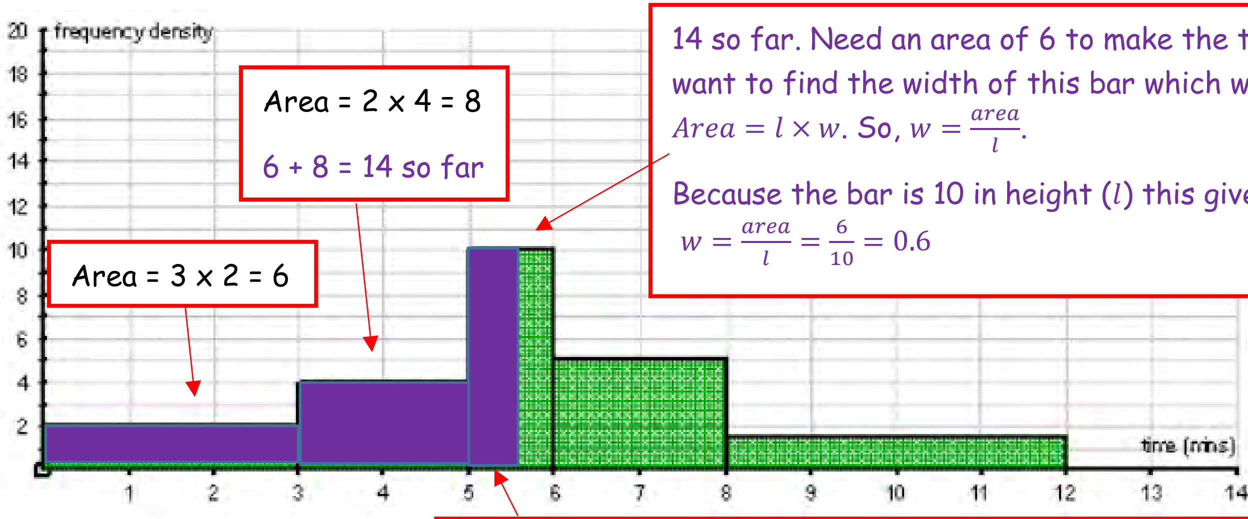
Finding the Median

The median is the middle point. We first need to work out how many Year 7's there were by adding up all the frequencies.

6 + 8 + 10 + 10 + 6 = 40.

Half of 40 is 20.

So, we need to find which point on the x-axis will give us an area of 20 up until that point.



14 so far. Need an area of 6 to make the total up to 20. We want to find the width of this bar which will give an area of 6. $Area = l \times w$. So, $w = \frac{area}{l}$.

Because the bar is 10 in height (l) this gives a width of: $w = \frac{area}{l} = \frac{6}{10} = 0.6$

The bar is 0.6 wide so this point must be 5.6. Therefore, **the median is 5.6.**

Mathematics

Higher

Unit 45

Inequalities <, >, ≤, ≥

What are Inequalities?

Inequalities are just another time-saving device and we use symbols instead of words.
They are a way of representing massive groups of numbers with just a couple of numbers and a fancy looking symbol.
For this topic, the prior knowledge that is needed is solving equations and drawing graphs,

What the inequality symbols mean

< means "is less than"

≤ means "is less than or equal to"

> means "is greater than"

≥ means "is greater than or equal to"

Always read from the symbol. This could mean that you read from right to left or left to right.

For Example:

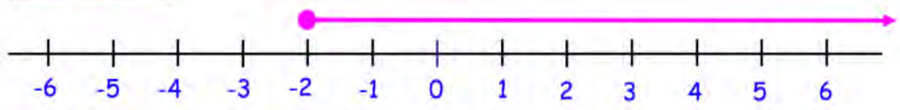
1.	$x < 5$	Means x is less than 5	So x could be 4, 0.6, -23... but NOT 5!
2.	$p \geq 100$	Means p is greater than or equal to 100	So p could be 104, 10000, 201.5... AND 100!
3.	$-2 < m$	Means x is greater than -2	So x could be -1.9, 0, 4.3... but NOT -2!

Representing Inequalities on a Number line

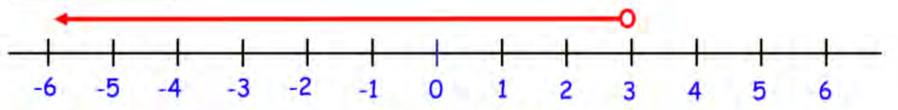
These are very common questions, and pretty easy ones too.

Method:
- Draw a line over all the numbers for which the inequality is true (the ones you can see, anyway)
- At the end of these lines, draw a circle, and colour it in if the inequality can equal the number, and leave it blank if it cannot.

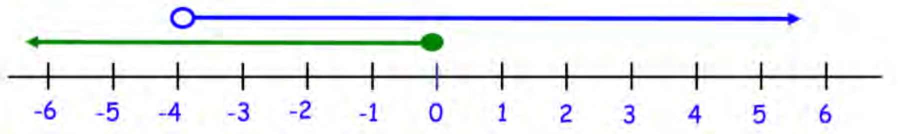
$x \geq -2$



$x < 3$



$x > -4$ and $x \leq 0$



Mathematics

Higher

Unit 45

Solving Inequalities

Important to know:

The rule for solving linear inequalities is exactly the same as that for solving linear equations, however

Just two things:

1. You cannot divide or multiply by a NEGATIVE value.

When dividing by the coefficient of x at the end of the inequality (like in equations) make sure it is a negative value first!

2. Do NOT ever use an equal (=) sign in your workings.

Example 1

$6x + 3 \geq 27$

$6x \geq 27 - 3$

$6x \geq 24$

$x \geq 24/6$

$x \geq 4$

Whole number values of x could be 4, 5, 6 ….

The smallest value that x could be is 4.

Example 2

$5x - 6 < 2x + 9$

$5x - 2x < 9 + 6$

$3x < 15$

$x < 15/3$

$x < 5$

Whole number values of x could be 4, 3, 2, 1,…..

The largest whole number value that x could be is 4.

For examples 1 & 2, the inequalities are solved using the same method that is used for solving equations but the inequality sign is kept throughout.

Example 3

$-2(5x - 4) > 98$

$-10x + 8 > 98$

$-10x > 98 - 8$

$-10x > 90$

$-90 > 10x$

$-90/10 > x$

$-9 > x$

This reads as x is less than -9.

The biggest whole number that x could be is -8.

For example 3, the inequality is solved using the same method that is used for solving equations with the inequality sign kept the same throughout. However, you cannot divide by a negative coefficient of x (-10) so you need to make it positive before dividing.

Example 4

$-4 < 3x + 5 \leq 2x + 8$

For double inequalities, one way to solve these is to split it into 2 parts to solve

$-4 < 3x + 5$ and then $3x + 5 \leq 2x + 8$

$-4 - 5 < 3x$ $\quad$ $3x - 2x \leq 8 - 5$

$-9 < 3x$ $\quad$ $x \leq 3$

$-9/3 < x$

$-3 < x$

This means that the value of x is greater than -3 but less than or equal to 3. This can be written as $-3 < x \leq 3$

Whole number values of x are -2, -1, 0, 1, 2, 3

Mathematics

Higher

Unit 45

Graphing Inequalities

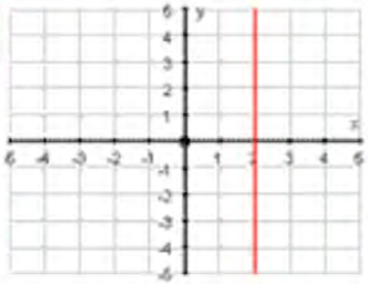
You are given one or more inequality and are asked to show the region on a graph which satisfies them all (i.e. every inequality works for every single point in your region).

You need to be able to draw straight line graphs for this topic.

e.g. 1 $x \leq 2$

1. Draw the line $x = 2$

2. Notice it is a solid line as x CAN be 2

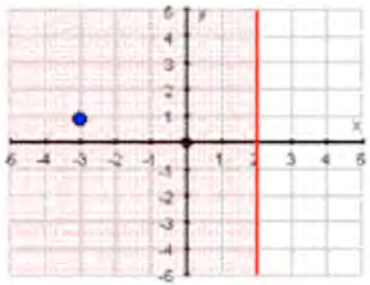


3. Choose a co-ordinate on one side of the line: e.g. (-3, 1).

$x = -3 \qquad y = 1$

$x \leq 2 \rightarrow -3 \leq 2$ √

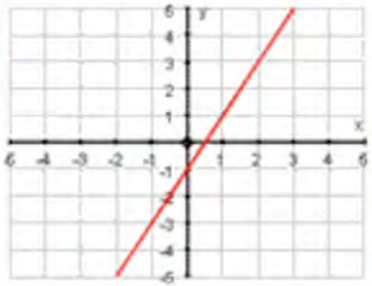
So our point is on the side of the line we want!



e.g. 2 $y > 2x - 1$

1. Draw the line $y = 2x - 1$

2. Notice it is a dashed line as y CANNOT be 2x - 1



3. Choose a co-ordinate on one side of the line: e.g. (4, 2).

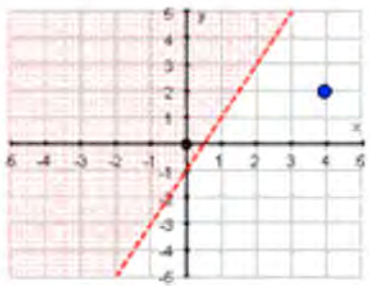
$x = 4 \qquad y = 2$

$y > 2x - 1$

$\rightarrow 4 > 8 - 1$

$\rightarrow 4 > 7$ ✗

So we want the other side of the line!



Method

1. Pretend the inequality sign is an equals sign and just draw your line

2. Look at the inequality sign and decide whether your line is dashed or solid

3. Pick a co-ordinate on either side of the line to help decide which region you want

Mathematics

Higher

Unit 45

e.g. 3 $x \geq 1$ $y > 2$ $5x + 8y \leq 40$

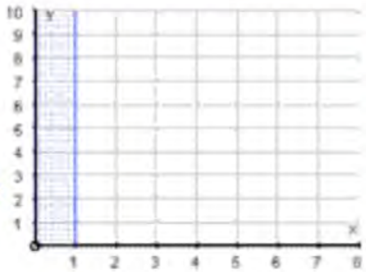
For questions like this, just deal with each inequality in turn, shading as you go!

Note: When you have got more than one inequality like this, it's normally best to shade the region you DON'T WANT, so you can leave the region you do want blank!

$x \geq 1$

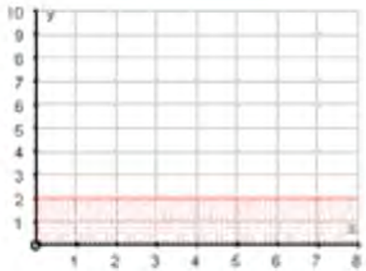
You should be able to do this one all in one.

The points where x is greater than 2 are to the right, so shade the left!



$y > 2$

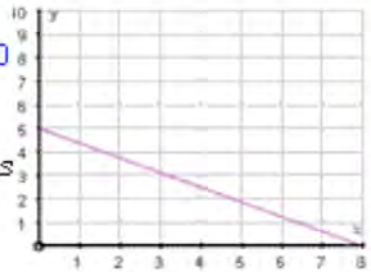
The big y values are all above the line, so let's shade the ones we don't want below the line!



$5x + 8y \leq 40$

1. Draw the line 5x+8y = 40

2. Notice it is a solid line as 5x+8y CAN be equal to 40



3. Choose a co-ordinate on one side of the line: e.g. (2, 1).

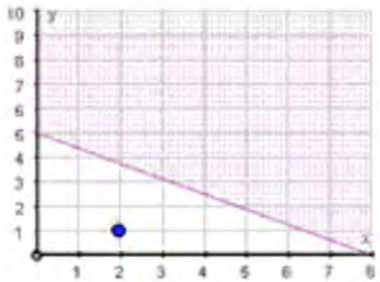
$x = 2$ $y = 1$

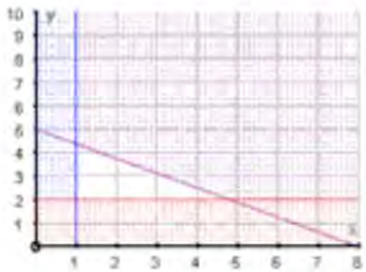
$5x + 8y < 40$

→ $10 + 8 < 40$

→ $18 > 7$ √

So we shade the other side





Putting it all together leaves us the blank region in the middle that satisfies all the inequalities!

The coordinates that lie in the region are:

(1, 3) (1, 4) (2, 3) (3, 3)

Method

1. Pretend the inequality sign is an equals sign and just draw EACH line.

2. Look at the inequality sign and decide whether your line is dashed or solid.

3. Pick a co-ordinate on either side of the line to help decide which region you want.

4. For questions that involve more than one inequality it's best to shade the region you DON'T WANT.

5. The coordinates that lie in the region are the ones within the required region and any ones that lie on the solid lines.

Mathematics

Higher

Unit 46

Direct & Indirect Proportion $\propto$ (Variation)

What does proportion mean and what symbol is used?

If two variables are proportional to each other, it just means that they are related to each other in a specific way.

The funny fish symbol $\propto$ just means "is proportional to".

There Two types of Proportion

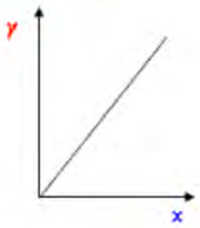
There are the two main types of proportion. These are direct proportion and indirect proportion.

(a) Direct Proportion

Both variables increase or decrease together

(i) Linear

Graph



Using symbols

$$y \propto x$$

y is proportional to x

y is directly proportional to x

y varies directly as x varies

Examples

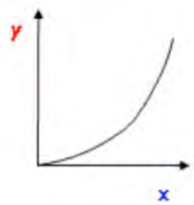
x could be the number of KitKat Chunkys that you buy

y could be the total cost of those KitKat Chunkys

As the number you buy increases, so too does the total cost

(ii) Quadratic

Graph



Using symbols

$$y \propto x^2$$

y is proportional to x^2

y is directly proportional to x^2

y varies directly as x^2 varies

Example

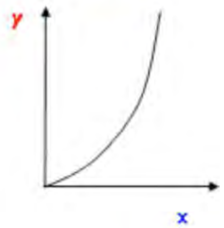
x could be the amount of money you spend advertising a gig

y could be the number of people who turn up to the gig

As the amount of advertising increases, word of mouth quickly spreads, and the number of people who go to the gig goes up by a lot.

(iii) Cubic

Graph



Using symbols

$$y \propto x^3$$

y is proportional to x^3

y is directly proportional to x^3

y varies directly as x^3 varies

Example

x could be the amount of time you spend on mrbartonmaths.com

y could be your maths exam mark

As the amount of time you spend revising on the site increases, everything begins to fall into place, and your marks just get higher and higher with each extra minute!

Mathematics

Higher

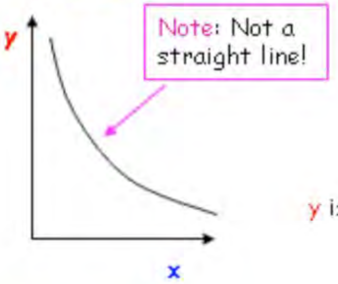
Unit 46

(b) Inverse Proportion

As one variable goes up, the other goes down

(i) Inverse

Graph



Using symbols

$$y \propto \frac{1}{x}$$

y is inversely proportional to x

Example

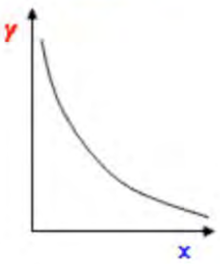
x could be the number of people you convince to join you on a road trip

y could be the amount each person must pay for petrol

As the number of people in the car increases, the amount everyone has to pay falls

(ii) Quadratic Inverse

Graph



Using symbols

$$y \propto \frac{1}{x^2}$$

y is inversely proportional to x^2

Example

x could be the number of hours you spend watching Big Brother

y could be your number of brain cells

As the hours increase, your brain cells disappear and an increasing rate!

How to tackle proportion questions

Whatever the question, whatever the type of proportion, this method will work.

Method

1. Decide on the type of Proportion
   - Direct or Indirect?
   - Linear, Quadratic or Cubic?
2. Write the expression with the funny fish sign
3. Make the expression into an equation by using = and k
4. Use the numbers they give you to find out the value of k
5. Write down the formula
6. Answer the questions!

Mathematics

Higher

Unit 46

Example 1

y is directly proportional to x. Given that y = 12 when x = 4, calculate the value of:
(a) y when x = 6 (b) x when y = 66

1. The question has told us that we are dealing with direct proportion, and unless it says otherwise, we can also assume that it is linear.
2. The expression to say that y is directly proportional to x is: $y \propto x$
3. Proportional means related to in a specific way. Once you decide what kind of proportion you are dealing with, all you need to do to get from x to y is to multiply by a number, which we call k.

Rule: Replace the $\propto$ sign with = and multiply the right hand side by k

$$y \propto x \longrightarrow y = kx$$

4. Now use the numbers in the question and put them in our formula… when y = 12, x = 4

$$12 = 4k$$

Rearranging gives us the value of k $\quad k = 12/4 = 3$

5. So now we have our formula: $y = 3x$
6. Now substitute in numbers and maybe (for b) rearrange.

(a) Find y when x = 6

$y = 3x \longrightarrow y = 3 \times 6$

So… $y = 18$

(b) Find x when y = 66

$y = 3x \longrightarrow x = y/3 \longrightarrow x = 66/3$

So… $x = 22$

Example 2

The variables p and q are related so that p is directly proportional to the square of q. Complete the table of values

p	0.5	2	
q		12	27

1. The question has told us that we are dealing with direct proportion, and it also mentions the word "square" which means we are dealing with quadratic.
2. The expression to say that p is directly proportional to the square of q is: $p \propto q^2$
3. Rule: Replace the $\propto$ sign with = and multiply the right hand side by k

$$p \propto q^2 \longrightarrow p = kq^2$$

4. The table tells us that when p = 2, q = 12. Put this information into the formula.

$$2 = 12^2 k \longrightarrow 2 = 144k$$

Rearranging gives us the value of k $\quad k = 2/144 = 1/72$

Note: a fraction is easier to use here

5. The formula is: $p = \frac{1}{72}q^2$
6. Now substitute in numbers and maybe (for b) rearrange.

(a) Find p when q = 27

$p = \frac{1}{72}q^2 \longrightarrow p = \frac{1}{72}27^2 \longrightarrow p = \frac{1}{72} \times 729$ So… $p = 10.125$

(b) Find q when p = 0.5

$p = \frac{1}{72}q^2 \longrightarrow q^2 = 72p \longrightarrow q^2 = 72 \times 0.5 \longrightarrow q^2 = 36$ So… $q = 6$

Example 3

z is inversely proportional to t. Given that when t = 0.3 the value of z = 16, find the value of z when t = 0.5

1. The question has told us that we are dealing with inverse proportion.
2. The expression to say that z is inversely proportional to t is: $z \propto \frac{1}{t}$
3. Rule: Replace the $\propto$ sign with = and multiply the right hand side by k, but be careful here!

$$z \propto \frac{1}{t} \longrightarrow z = \frac{k}{t}$$

Note: We are multiplying by k, so it goes on the top!

4. Now use the numbers in the question and put them in our formula… when z = 16, t = 0.3

$$16 = \frac{k}{0.3}$$

Rearranging gives us the value of k $\quad k = 0.3 \times 16 = 4.8$

5. The formula is: $z = \frac{4.8}{t}$
6. Now substitute in numbers:

Find z when t = 0.5

$z = \frac{4.8}{t} \longrightarrow z = \frac{4.8}{0.5} \longrightarrow$ So… $z = 9.6$

Mathematics

Higher

Unit 47

Non-Right-Angled Triangles

For triangles that are not right-angled ones we cannot use Pythagoras' Theorem or SOHCAHTOA Trigonometry.

This means that we use further trigonometry methods called the Sine and Cosine rules.

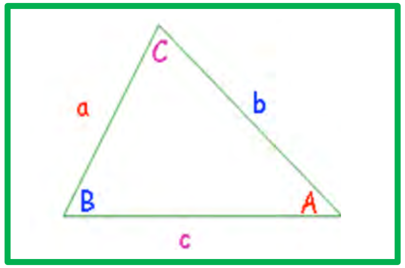
These rules work for any type of triangle as long as you have the correct information to use each of the rules.

The Crucial Point about the Sine and Cosine Rules

You must know when to use each rule and what information you need to be given.

Then it's just plugging numbers into formulas!

Note: In all the formulas, small letters represent sides, and Capital Letters represent Angles!



The Cosine Rule – Finding an unknown Side

What Information do you need to be given?

Two sides of the triangle and the INCLUDED ANGLE (i.e. the angle between the two sides!)

What is the Formula?

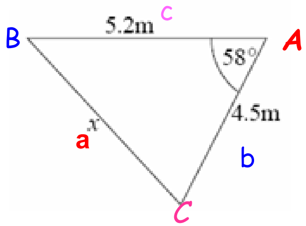
$$a^2 = b^2 + c^2 - 2bcCosA$$

Remember:

The small letters represent sides, and Capital Letters represent Angles!

You need to take care when you input the values into your calculator.

Example: Find the length of x



$a^2 = b^2 + c^2 - 2bcCosA$

$a^2 = 5.2^2 + 4.5^2 - 2 \times 5.2 \times 4.5 \times Cos58$

$a^2 = 5.2^2 + 4.5^2 - 2 \times 5.2 \times 4.5 \times Cos58$

$a^2 = 22.48977...$

$a = \sqrt{22.48977...}$ ← Square root

$a = 4.742338...$

$\therefore x = 4.74m$ (2dp)

Method

- Label the side you want to find as 'a'
- This means the angle opposite it is 'A'
- Label the other 2 sides as 'b' and 'c'
- Label the opposite angles to 'b' and 'c' as 'B' and 'C'
- Write down the formula $a^2 = b^2 + c^2 - 2bcCosA$
- Substitute the numbers into the correct places for the letters
- Remember that 2bcCosA means 2 × b × c × Cos A
- Use your calculator to work out the value of a^2
- Square root this number to find the value of 'a'. This is then the value of 'x' in this question.
- You may also be asked to round your answer to a specific amount.

Mathematics

Higher

Unit 47

$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

The Cosine Rule – Finding an unknown Angle

What Information do you need to be given?

All three lengths of the triangle must be given!

What is the Formula?

$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

Remember:

This is just a re-arrangement of

$a^2 = b^2 + c^2 - 2bcCosA.$

In the exam you will only be given

$a^2 = b^2 + c^2 - 2bcCosA.$

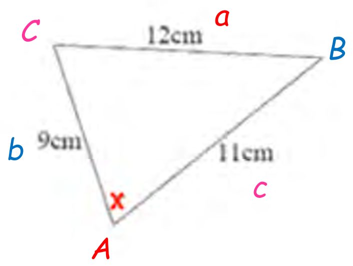
Method

- Label the angle you want to find as 'A'
- This means the side opposite it is 'a'
- Label the other 2 angles as 'B' and 'C'
- Label the opposite sides to 'B' and 'C' as 'b' and 'c'
- Write down the formula

$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

- Substitute the numbers into the correct places for the letters
- Remember that 2bc means 2 × b × c
- Calculate the value of the top of the formula and calculate the value of the bottom of the formula
- If the value of the top is a negative one, it means the size of the angle should be an obtuse one. If it's positive it will be an acute angle
- Use $Cos^{-1}(58/198)$ to find the size of the angle. This is a similar method to when finding an angle using SOHCAHTOA
- You may also be asked to round your answer to a specific amount.

Example



$$CosA = \frac{b^2 + c^2 - a^2}{2bc}$$

$$Cosx = \frac{9^2 + 11^2 - 12^2}{2 \times 9 \times 11}$$

$$Cosx = \frac{58}{198} \rightarrow Cos^{-1}(58/198)$$

$$Cosx = 0.292929\ldots \rightarrow x = 72.97^0 (2dp)$$

Mathematics

Higher

Unit 47

Remember:

For the Sine rule, you place the small letters which represent sides on the top when finding a side and place the Capital Letters which represent Angles on the top when finding an Angle.

Also think of the Sine Rule as pairs!

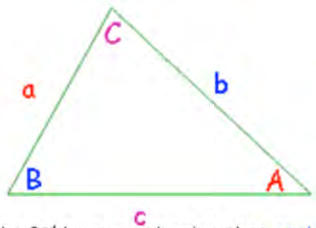
The Sine Rule – Finding an unknown Side

What Information do you need to be given?

Two angles and the length of a side

What is the Formula?

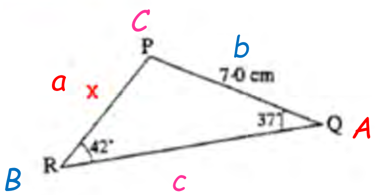
$$\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC}$$



Remember:

If you are given two angles, you can easily work out the 3rd by remembering that angles in a triangle add up to 180⁰!

Example



$$\frac{a}{SinA} = \frac{b}{SinB}$$

$$\frac{x}{Sin37} = \frac{7.0}{Sin42}$$

$$x = \frac{7.0}{Sin42} \times Sin37$$

$$x = 6.3cm \text{ (1dp)}$$

The Sine Rule – Finding an unknown Angle

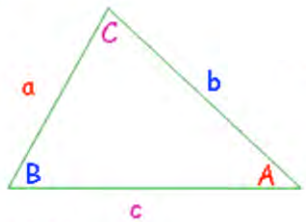
What Information do you need to be given?

Two lengths of sides and the angle NOT INCLUDED

(i.e. not between those two sides!)

What is the Formula?

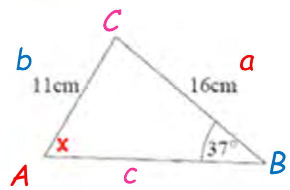
$$\frac{SinA}{a} = \frac{SinB}{b} = \frac{SinC}{c}$$



Remember:

If the angle is included, you will have to use the Cosine Rule!

Example



$$\frac{SinA}{a} = \frac{SinB}{b}$$

$$\frac{Sinx}{16} = \frac{Sin37}{11}$$

$$Sinx = \frac{Sin37}{11} \times 16$$

Use $Sin^{-1}(0.8753)$ → $Sinx = 0.8753... \rightarrow x = 61.1^0 \text{ (1dp)}$

Method

- Label the side you want to find as 'a'. This means the angle opposite it is 'A'
- Label the other side that you know as 'b' and its opposite angle as 'B'. You do not need to label the other side and angle unless you want to or if you need to use them
- Write down the formula $\frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC}$
- You only need to use 2 of the pairs
- Substitute in what you know into the correct place
- Rearrange to get 'a' on its own. This will be the value of x.
- You may also be asked to round your answer to a specific amount.

Method

- Label the Angle you want to find as 'A'. This means the side opposite it is 'a'
- Label the other Angle that you know as 'B' and its opposite angle as 'b'. You do not need to label the other angle and side unless you want to or if you need to use them
- Write down the formula $\frac{SinA}{a} = \frac{SinB}{b} = \frac{SinC}{c}$
- Again, you only need to use 2 of the pairs
- Substitute in what you know into the correct place
- Rearrange to get 'Sin A' on its own.
- To find 'A' you will need to use $Sin^{-1}(0.8753)$
- You may also be asked to round your answer to a specific amount.

Mathematics

Higher

Unit 47

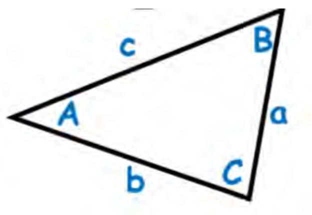
Area of a triangle where you DO NOT know the perpendicular height

What is the Formula?

The formula to find the area of a triangle when the perpendicular height is not known is:

$$Area = \frac{1}{2}ab\sin c$$

The letters in this formula relate to the triangle labelled in the same way as the Sine and Cosine rules.

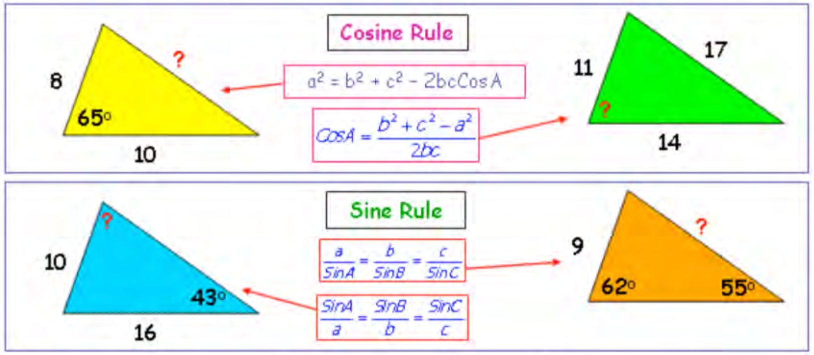


What Information do you need to be given?

Two sides of the triangle and the INCLUDED ANGLE

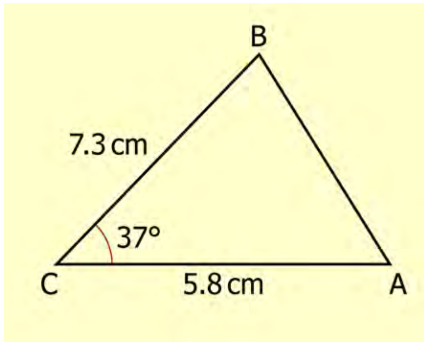
(i.e. the angle between the two sides!)

A Summary of the Sine and Cosine rules



	Finding Sides	Finding Angles
Cosine Rule	Need 2 sides and included angle	Need all 3 sides
Sine Rule	Need 2 angles and any side	Need 2 sides and an angle not included

Example: Find the area of the triangle



$$Area = \frac{1}{2}ab\sin c$$

Area = $\frac{1}{2}$ × 7.3 × 5.8 × sin 37

Area = 12.74 cm^2 (2dp)

Method

- Label the included angle between the 2 sides that you know as 'C'
- This means the side opposite it is 'c'
- Label the other 2 sides as 'b' and 'c'
- Label the opposite angles to 'b' and 'c' as 'B' and 'C'
- Write down the formula

  $$Area = \frac{1}{2}ab\sin c$$
- Substitute the numbers into the correct places for the letters and type into the calculator
- You can use 1 ÷ 2 or 0.5 for $\frac{1}{2}$ on your calculator
- You may also be asked to round your answer to a specific amount.

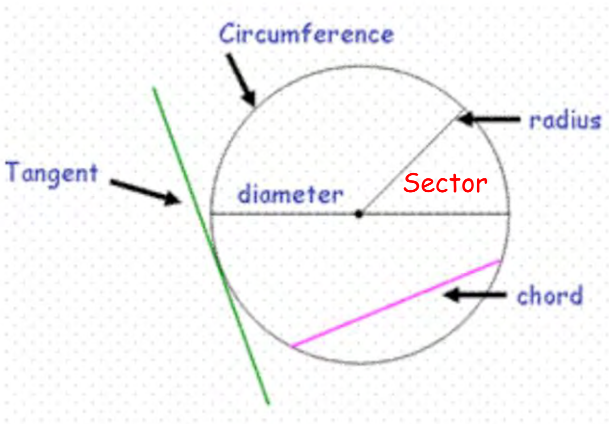
Mathematics

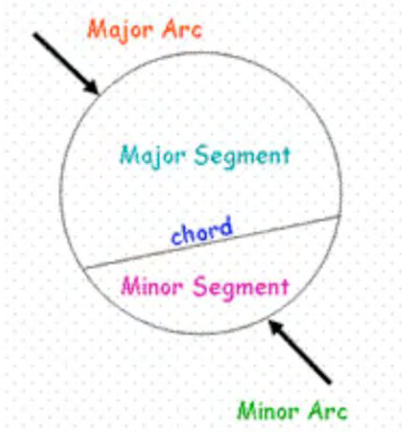
Higher

Unit 48

Arcs, Sectors & Segments

Parts of a circle





Parts of a circle

The **radius** is the line from the centre to the outside of the circle.

The **diameter** is the line that goes from one side of the circle to the other and passes through the centre.

The **tangent** is a straight line that touches the outside of the circle at one point.

A **chord** is a straight line that goes from one side of the circle to the other but does not pass through the centre.

The **circumference** is the distance around the outside of a circle.

An **arc** is part (or fraction) of the circumference of a circle. We can have a major (large) arc and a minor (small) arc.

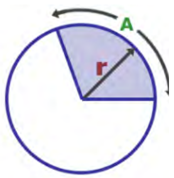
A **sector** is the area between two radii. It is a part (or fraction) of the area of a circle. We can have a major sector and a minor sector.

A **segment** is the area between a chord and an arc. We can have a major segment and a minor segment.

Length of an arc

As an **arc** is a fraction of the circumference of a circle, we use the formula of the circumference of a circle within the formula for length of an arc. θ is the angle at the centre.

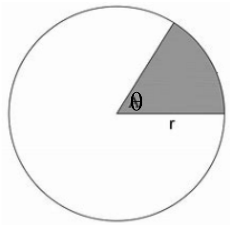
Length of arc = $\frac{\theta}{360}\pi d$.



Area of a sector

As a **sector** is a fraction of the area of a circle, we use the formula of the area of a circle within the formula for area of a sector. θ is the angle at the centre.

Area of sector = $\frac{\theta}{360}\pi r^2$.

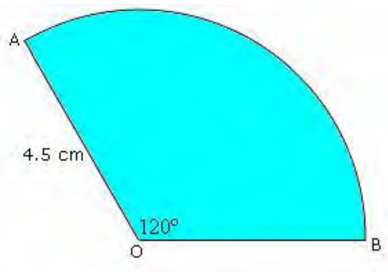


Mathematics

Higher

Unit 48

Example 1: Find the length of the arc AB

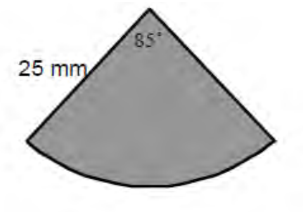


Length of arc = $\frac{\theta}{360} \times \pi D$

$$= \frac{120}{360} \times \pi \times 9$$

= 9.42cm to 2dp

Example 2: Find the area of the shaded sector

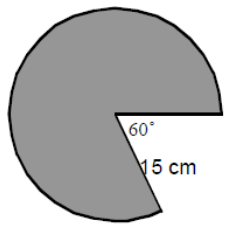


Area of sector = $\frac{\boldsymbol{\theta}}{\mathbf{360}} \boldsymbol{\pi r^2}$

$$= \frac{85}{360} \times \pi \times 25^2$$

= 463.6cm^2 to 1dp

Example 3: Find the area of the shaded sector



Area of sector = $\frac{\boldsymbol{\theta}}{\mathbf{360}} \boldsymbol{\pi r^2}$

$$= \frac{300}{360} \times \pi \times 15^2$$

= 589cm^2 to 3sf

Method for example 1

- Read the question carefully and decide which formula to use
- Write down the formula

  Length of arc = $\frac{\theta}{360} \times \pi D$
- Substitute the numbers into the correct places for the letters and type into the calculator
- You may also be asked to round your answer to a specific amount.

Method for examples 2 & 3

- Read the question carefully and decide which formula to use
- Write down the formula

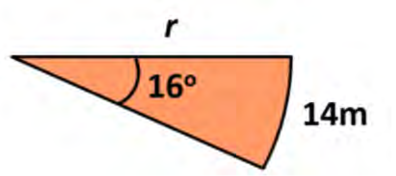
  Area of sector = $\frac{\theta}{360} \times \pi r^2$
- Substitute the numbers into the correct places for the letters and type into the calculator
- For example 3 the sector angle is not 60° but 300°
- You may also be asked to round your answer to a specific amount.

Mathematics

Higher

Unit 48

Example 3: Find the length of the radius r.



Length of arc = $\frac{\theta}{360} \times \pi D$

$$14 = \frac{16}{360} \times \pi \times D$$

14 ÷ 0.13962… = D

100.2676141 = D

∴ r = 100.2676141 ÷ 2

= 50.1cm 1dp

Method for example 3

- Read the question carefully and decide which formula to use
- We are given the length of the arc and need to find the radius so write down the formula:

  Length of arc = $\frac{\theta}{360} \times \pi D$
- Substitute the numbers into the **correct** places for the letters
- Calculate what you can before rearranging as this is easier. If you are able to rearrange before doing any calculations be careful that you have done it correctly
- Because the formula involves the diameter and we need to find the radius divide the answer from using the formula by 2
- You may also be asked to round your answer to a specific amount.

Example 4: Find the sector angle

Area of sector = $\frac{\boldsymbol{\theta}}{\mathbf{360}} \boldsymbol{\pi} r^2$

$$120 = \frac{\theta}{360} \times \pi \times 14^2$$

$$120 = \theta \times 1.71042 ....$$

120 ÷ 1.71042 = θ

∴ θ = 70.2° to 1dp

Method for example 4

- Read the question carefully and decide which formula to use
- We are given the area of the sector and need to find the sector angle so write down the formula:

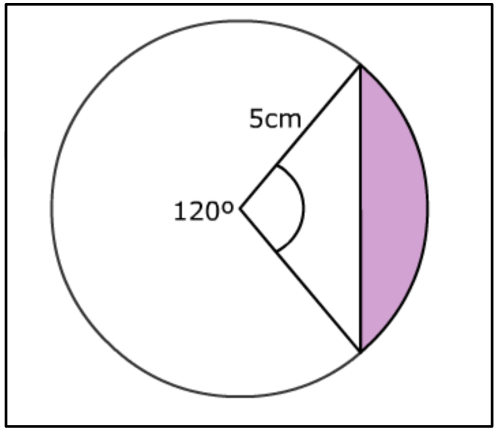
  Area of sector = $\frac{\theta}{360} \times \pi r^2$
- Substitute the numbers into the **correct** places for the letters
- Calculate what you can before rearranging as this is easier. If you are able to rearrange before doing any calculations be careful that you have done it correctly
- You may also be asked to round your answer to a specific amount.

Mathematics

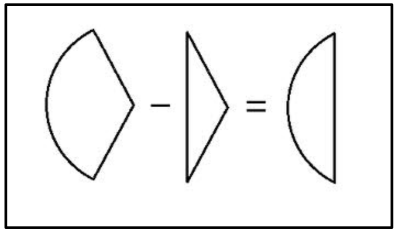
Higher

Unit 48

Area of a segment (or area of shaded part)



HINT



Method to find Area of a segment

Area of segment = Area of sector – Area of triangle

Area of segment = $\frac{\theta}{360}\pi r^2$ - $\frac{1}{2}$ $ab\ \sin C$

Workings for question above

Area of segment = Area of sector – Area of triangle

Area of segment = $\frac{\theta}{360}\pi r^2$ - $\frac{1}{2}$ $ab\ \sin C$

Area of segment = $\frac{120}{360} \times \pi \times 5^2$ - $\frac{1}{2}$ × 5 × 5 × sin 120

Area of segment = 26.1799…. – 10.8253…..

Area of segment = 15.35 cm^2 (2dp)

Method to find Area of a segment

- Read the question carefully and decide which method and formulae you need to use
- Write the formulae down. You can either do area of a sector and the area of a triangle separately and then subtract the answers or do it as one method as shown.

  Area of segment = $\frac{\theta}{360}\pi r^2$ - $\frac{1}{2}$ $ab\ \sin C$
- Substitute the numbers into the **correct** places for the letters
- Calculate each part and then subtract these answers. Remember the area of a triangle requires you to label the triangle with a, b, c, A, B, & C or you will remember to use the 2 sides and the angle between them. You may need to look back at Unit 48.
- You may also be asked to round your answer to a specific amount.

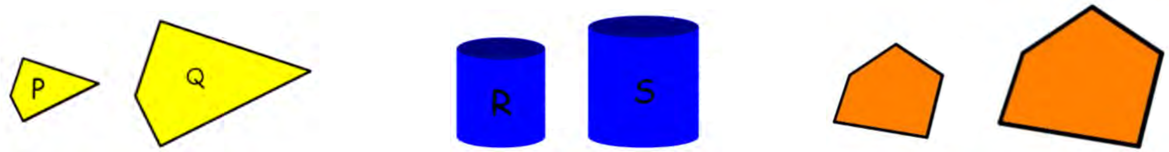
Mathematics

Higher

Unit 49

Similar Shapes

Shapes are classed as similar if they are the same shape and one of them is an **enlargement** of the other.



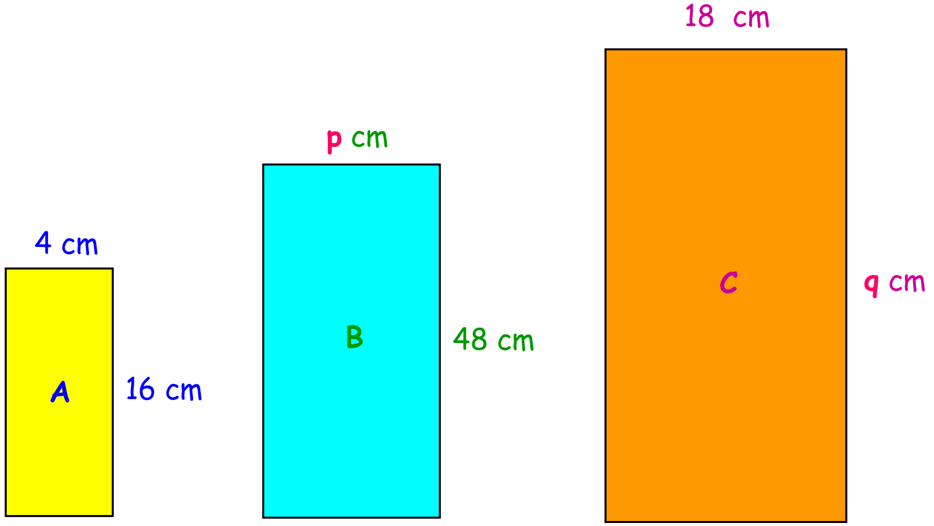
Technically, to get from one object to the other you must multiply (or divide) **every single length** by the same number

This number is called the **Scale Factor**.

Using Length Scale Factors

If we are told that two objects are similar, and we can work out the scale factor, then it is possible to work out a lot of unknown information about both objects

Example: - These three shapes are similar. Find the missing values



To Find p:

Work out the scale factor between rectangles A and B:

$$48 \div 16 = 3$$

So, every length on Rectangle A is enlarged by a scale factor of 3 to get the lengths of Rectangle B.

$$p = 4 \times 3$$

$$= 12cm$$

To Find q:

Work out the scale factor between rectangles A and C:

$$18 \div 4 = 4.5$$

So, every length on Rectangle A is enlarged by a scale factor of 4.5 to get the lengths of Rectangle C.

$$q = 16 \times 4.5$$

$$= 72cm$$

Mathematics

Higher

Unit 49

Similar Triangles

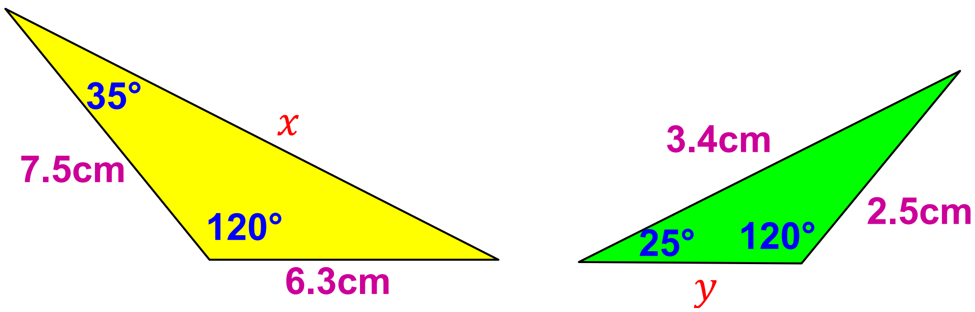
For any other shape to be similar, all angles must be the same and all matching sides must be in proportion.

But all you need for similarity between two **triangles** is for **all three angles to be the same.** Then you can be sure one triangle is an enlargement of the other.

Example:

(a) How do you know these two triangles are similar?

(b) Find the unknown lengths



(a) Two triangles are similar if all their angles are the same.

We can work out the missing angles in each triangle using the fact that angles in a triangle add to 180°.

The missing angle in the yellow triangle is:

180 – (120 + 35) = 25°

The missing angle in the green triangle is:

180 – (120 + 25) = 35°.

All the angles are the same, so the triangles are similar.

(b) As the triangles are similar, we can work out the scale factor, using our matching sides between the 120° and the 35°.

$$7.5 \div 2.5 = 3$$

So, to get from one triangle to the other, we either multiply or divide by 3.

Therefore, $x = 3.4 \times 3$ $\quad$ $y = 6.3 \div 3$

$= 10.2cm$ $\quad$ $= 2.1cm$

Mathematics

Higher

Unit 49

Area and Volume Scale Factors

It is also possible for 3D shapes to be similar.

If we know the scale factor between the lengths of sides, we can also say that:

Area scale factor = (scale factor of lengths)^2

Volume scale factor = (scale factor of lengths)^3

Example 1: These two containers are similar. Work out the volume of water the smaller one can hold.

First, work out the length scale factor in exactly the same way as we did for similar shapes/triangles with corresponding sides.

$$60 \div 40 = 1.5$$

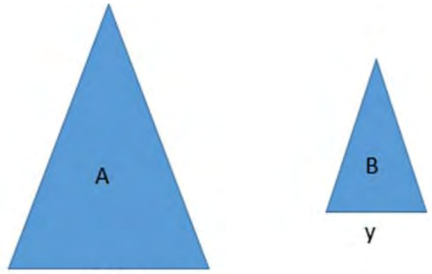
If our length scale factor = 1.5

The volume scale factor = 1.5^3 = 3.375

Now we know the volume scale factor,

Volume of the small container: $20.25 \div 3.375 = 6$ litres

Example 2: Triangle A and triangle B are similar. The area of A is 250cm^2 and the area of B is 10cm^2. Find the length y.



To find the scale factor for <u>area</u> we divide the area of A by the area of B.

Scale factor for <u>area</u> = 250 ÷ 10 = 25

To find the scale factor for length, we need to square root this.

Scale factor for length = $\sqrt{25} = 5$

To find missing lengths we multiply or divide by 5. (In this case, divide because we are finding the length on the smaller shape.)

Y = 100 ÷ 5 = 20cm

Mathematics

Higher

Unit 50

Error Approximation / Limits of Accuracy

Error approximation / limits of accuracy are the upper/greatest and lower/least bounds of a number before it was rounded.

The general rule for finding the upper and lower bounds is:

"If you are measuring to the **nearest** x cm, the greatest and least measurement will be half of x above and below the rounded answer"

Example 1:

Key word

The length of a book is 5cm to the **nearest** centimetre, find the upper and lower bounds for the length of the book.

Step 1: Half the units (the number after the word "**nearest**").

Half of 1cm is 0.5cm.

Step 2: Take this away from the original length to find the lower bound.

5 – 0.5 = 4.5cm

Add this to the original length to find the upper bound.

5 + 0.5 = 5.5cm

Lower bound = 4.5cm, upper bound = 5.5cm

Example 2:

The length of a room is 50m to the **nearest** 10 metres, find the upper and lower bounds for the length of the room.

Step 1: Half the units (the number after the word "**nearest**").

Half of 10m is 5m.

Step 2: Take this away from the original length to find the lower bound.

50 – 5 = 45m

Add this to the original length to find the upper bound.

50 + 5 = 55m

Lower bound = 45m, upper bound = 55m

Example 3:

A paving slab measures 60cm to the **nearest** 10 centimetres, what is the least and greatest length of:

a) One paving slab.

Step 1: Half the units (the number after the word "**nearest**").

Half of 10cm is 5cm.

Step 2: Take this away from the original length to find the lower bound / the least length. 60 – 5 = 55cm

Add this to the original length to find the upper bound / the greatest length 60 + 5 = 65cm

Least length = 55cm, greatest length = 65cm

b) A path of 40 paving slabs?

Least length

55 x 40 = 2,200cm

Greatest length

65 x 40 = 2,600cm

Mathematics

Higher

Unit 50

Upper and lower bounds with decimal places.

If a number has been written as 5.6 correct to one decimal place, then the true value of the number lies between 5.55 and 5.65.

Examples:

Write the upper and lower bounds for the following numbers for the degree of accuracy given.

a) 5.69 to 2 d.p.

2 d.p. is the value 0.01, half of this is 0.005.
The lower bound is 5.69 – 0.005 = 5.685.
The upper bound is 5.69 + 0.005 = 5.695.

b) 56.43 to 2 d.p.

2 d.p. is the value 0.01, half of this is 0.005.
The lower bound is 56.43 – 0.005 = 56.425.
The upper bound is 56.43 + 0.005 = 56.435.

c) 45.356 to 3 d.p.

3 d.p. is the value 0.001, half of this is 0.0005.
The lower bound is 45.356 – 0.0005 = 45.3555.
The upper bound is 45.356 + 0.0005 = 45.3565.

Upper and lower bounds with significant figures.

If a number has been written as 3.44 correct to three significant figures, then the true value of the number lies between 3.435 and 3.445.

Examples:

Write the upper and lower bounds for the following numbers for the degree of accuracy given.

a) 4 to 1 s.f.

1 s.f. is the value 1, half of this is 0.5.
The lower bound is 4 – 0.5 = 3.5.
The upper bound is 4 + 0.5 = 4.5.

b) 23 to 2 s.f.

2 s.f. is the value 1, half of this is 0.5.
The lower bound is 23 – 0.5 = 22.5.
The upper bound is 23 + 0.5 = 23.5.

c) 35.4 to 3 s.f.

3 s.f. is the value 0.1, half of this is 0.05.
The lower bound is 3 – 0.05 = 35.35.
The upper bound is 3 + 0.05 = 35.45.

Mathematics

Higher

Unit 50

Adding Measures.

When a calculation involves adding two or more measurements together (a + b)
The lower bound is found by:
Adding the lower bounds together ($a_{min} + b_{min}$)
The upper bound is found by:
Adding the upper bounds together ($a_{max} + b_{max}$)

Example:

4 boxes have been stacked on top of each other. Two of the boxes have heights of 25cm, correct to the **nearest** 5cm. The other two boxes have heights of 10cm, correct to the **nearest** 5cm. What is the least and greatest possible height of the stack of boxes?

Step 1: Half the units (the number after the word "**nearest**").
Half of 5cm is 2.5cm.
Step 2: Take this away from the original heights to find the lower bounds.
For the 25cm boxes, 25 – 2.5 = 22.5cm
For the 10cm boxes, 10 – 2.5 = 7.5cm

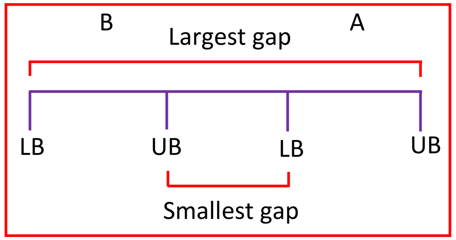
Add this to the original heights to find the upper bounds.
For the 25cm boxes, 25 + 2.5 = 27.5cm
For the 10cm boxes, 10 + 2.5 = 12.5cm

Lower bound / least height = 22.5 + 22.5 + 7.5 + 7.5 = 60cm

Upper bound / greatest height = 27.5 + 27.5 + 12.5 + 12.5 = 80cm

Subtracting Measures.

When a calculation involves subtracting two measurements (a – b)
The lower bound is found by:
Subtracting the upper bound from the lower bound ($a_{min} - b_{max}$)
We want the smallest / least gap
The upper bound is found by:
Subtracting the lower bound from the upper bound ($a_{max} - b_{min}$)
We want the largest / greatest gap



Example:

Number A is given as 36 to the **nearest** whole number and number B is given as 23 to the nearest whole number. Find the lowest and greatest possible values for A – B.

Step 1: Half the units (the number after the word "**nearest**").
Half of 1 is 0.5.
Step 2: Take this away from the original numbers to find the lower bounds.
For A, 36 – 0.5 = 35.5
For B, 23 – 0.5 = 22.5cm

Add this to the original numbers to find the upper bounds.
For A, 36 + 0.5 = 36.5
For B, 23 + 0.5 = 23.5

Lower bound / least value for A – B = lower bound of A – upper bound of B
35.5 – 23.5 = 12

Upper bound / greatest value for A – B = upper bound of A – lower bound of B
36.5 – 22.5 = 14

Mathematics

Higher

Unit 50

Problem Solving Examples

Example 1:

The capacity of a jug is 250ml, measured to the **nearest** 10ml.
a) Write down the least and greatest value of the capacity of the jug.

Step 1: Half the units (the number after the word "**nearest**").
Half of 10ml is 5ml.

Step 2: Take this away from the original capacity to find the lower bound / least value of one jug. 250 – 5 = 245ml
Add this to the original capacity to find the upper bound / greatest value of one jug. 250 + 5 = 255ml

b) The capacity of a bucket is 5.1 litres, measured correct to the **nearest** $\frac{1}{10}$ of a litre. The jug is filled with water and then the water is poured into the bucket. This is done 20 times in all. Explain, showing all your calculations, why it is not always possible for the bucket to hold all this water.

Step 1: Half the units (the number after the word "**nearest**").
Half of 0.1 litres is 0.05 litres.

Step 2: Take this away from the original capacity to find the lower bound of the bucket. 5.1 – 0.05 = 5.05 litres
Add this to the original capacity to find the upper bound of the bucket.
5.1 + 0.05 = 5.15 litres

Least amount for 20 jugs	Greatest amount for 20 jugs	Least capacity of bucket	Greatest capacity of bucket
245 x 20 = 4,900ml	255 x 20 = 5,100ml	5.05 litres = 5,050ml	5.15 litres = 5150ml

The greatest amount in 20 jugs (5, 100ml) would overflow in the least possible capacity for the bucket (5.05 litres / 5,050ml).

Example 2:

James is stacking 5 boxes in his garage.

The height of the garage is 2.6m correct to the **nearest** 10cm. (2.6m = 260cm)

The heights of the 5 boxes are 50cm correct to the **nearest** 5cm.

Calculate the maximum possible gap between the stack of 8 boxes and the garage ceiling.

We want the largest / greatest gap ($garage_{max}$ – $boxes_{min}$)

Minimum height of the boxes
Step 1: Half the units (the number after the word "**nearest**").
Half of 5cm is 2.5cm.
Step 2: Take this away from the original height to find the lower bound of one box
50 – 2.5 = 47.5cm
Step 3: Multiply by the number of boxes to find the lower bound for the stack of boxes.
47.5 x 5 = 237.5cm

Maximum height of the garage
Step 1: Half the units (the number after the word "**nearest**").
Half of 10cm is 5cm.
Step 2: Add this to the original height to find the upper bound of the garage
260 + 5 = 265.5cm

Greatest gap = 265.5 – 237.5 = 28cm

Mathematics

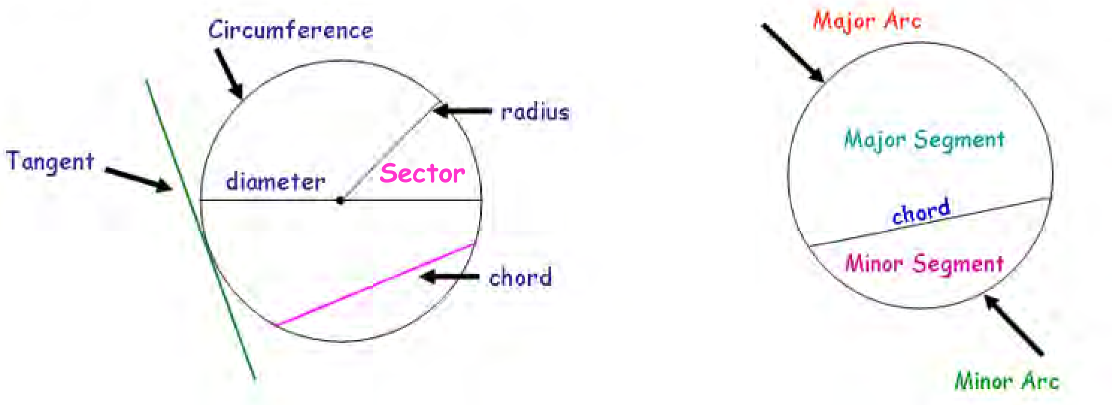
Higher

Unit 51

Circle Theorems

Parts of a Circle

It is important we know the names for each part of the circle before we start looking at circle theorems.



Tips for Answering Circle Theorem Questions

1. Always write down the name of each of the Circle Theorems you have used to get your answer (even if there are more than one)

2. An angle is **not a right-angle** just because it looks like one. You must be able to prove it using a circle theorem or be told it in the question.

3. You will also need to use other angle facts to be able to answer circle theorem questions (See Unit 03 for a recap).

4. Often there are lots of different ways of working out the answer

Three things you should Learn about Circle Theorems:

1) What each of the theorems say
2) How to spot them
3) How to show you are using circle theorems in your answers

Mathematics

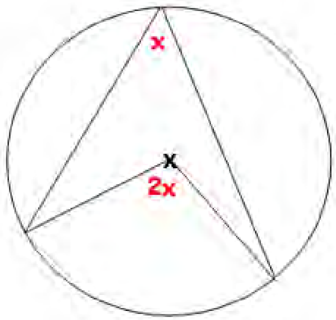
Higher

Unit 51

Theorem 1: Angle at the Centre

Fact: The angle at the centre is twice as big as the angle at the circumference made by the same arc or chord

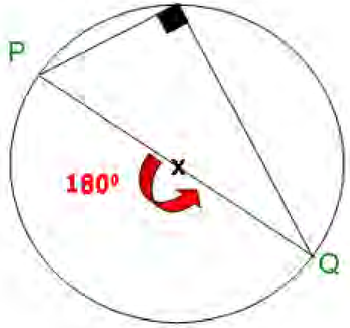
How to spot it: Start with two points (could be the ends of a chord). If you go point-centre-point, the angle you make will be twice as big as if you go point-circumference-point



Theorem 2: Angles in a Semi-Circle

Fact: The angle made at the circumference in a semi circle is a right angle (90°)

How to spot it: Look for a triangle whose base is the diameter of the circle (a line going through the centre). The angle at the circumference in this triangle will always be a right angle

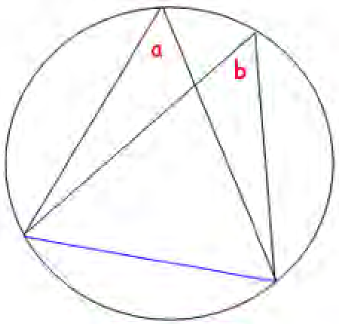


Theorem 3: Angles in the Same Segment

(Or Angles subtended on the same arc)

Fact: Angles in the same segment of a circle are equal to each other

How to spot it: Start with two points (could be the ends of a chord). If you go point-circumference-point, the angle you make will be exactly the same as if you go point-circumference-point, so long as you stay in the same segment of the circle.



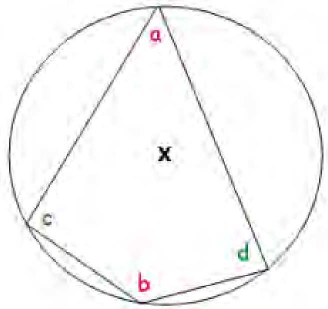
$a = b$

Theorem 4: Cyclic Quadrilateral

Fact: The opposite angles in a cyclic quadrilateral add up to 180^0

How to spot it: Look for a four-sided shape with each of the corners on the circumference. The opposite angles in this shape will always add up to 180°

Note: Just like any other quadrilateral, the sum of all the interior angles is still 360°



$a + b = 180^0$

$c + d = 180^0$

Mathematics

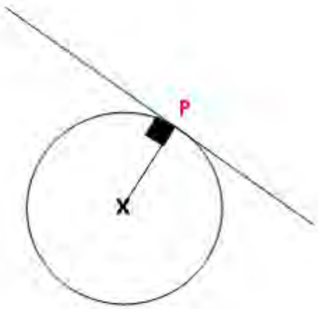
Higher

Unit 51

Theorem 5: Tangent

Fact: The angle made by a tangent and the radius is a right-angle (90°)

How to spot it: A tangent is a straight line that only touches a circle in one place. If you draw a line from that one place to the centre of a circle, then the angle you form is always a right-angle.

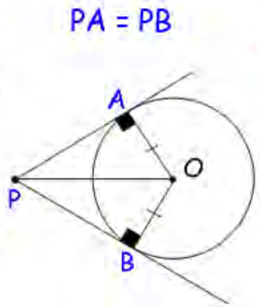


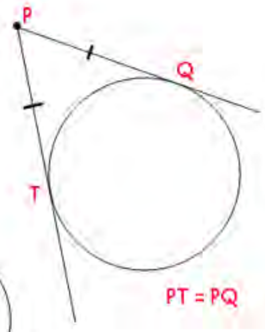
Theorem 6: Two Tangents

Fact: From any point outside the circle, you can only draw two tangents to the circle, and these tangents will be equal in length.

How to spot it: Look for where the tangents to a circle meet. The lengths between where they touch the circle and the point at which they meet will always be the same

Note: More often than not, this theorem leads to some isosceles triangles, so be on the look out.

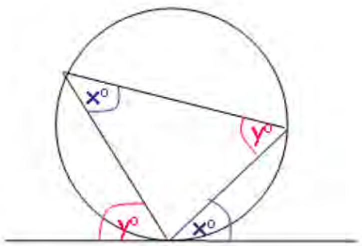




Theorem 7: Alternate Segment Theorem

Fact: The angle between a tangent and a chord at the point of contact is equal to the angle made by that chord in the other segment of the circle.

How to spot it: Look for a tangent and a chord meeting at the same point. The angle they make is exactly the same as the angle at the circumference made by that chord - imagine the chord is the base of a triangle, and the angle you want is at the top of the triangle.

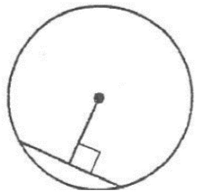


Theorem 8: A radius always meets a chord at a right angle.

Fact: The perpendicular bisector of any chord passes through the centre of the circle.

How to spot it: The perpendicular bisector of any chord passes through the centre of the circle.

Extra fact: Two radii can create an isosceles triangle

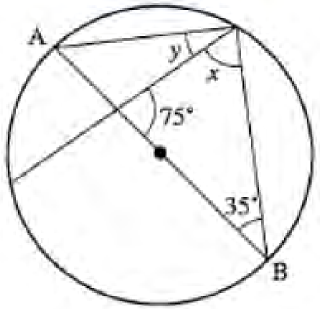


Mathematics

Higher

Unit 51

Example 1:



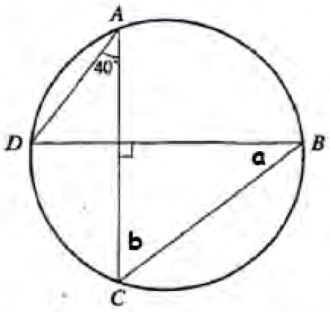
x = 180 - 75 - 35 = 70°

(angles in a triangle)

y = 90 - 70 = 20°

(Theorem 2 - angles in a semi-circle)

Example 2:



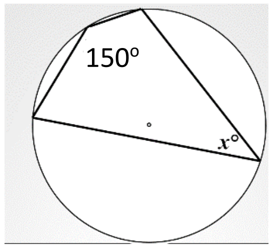
a = 40°

(Theorem 3 - angles in the same segment)

b = 180 - 90 - 40 = 50°

(angles in a triangle)

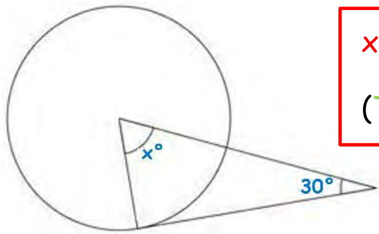
Example 3:



x = 180 – 150 = 30°

(Theorem 4 – angles in a cyclic quadrilateral)

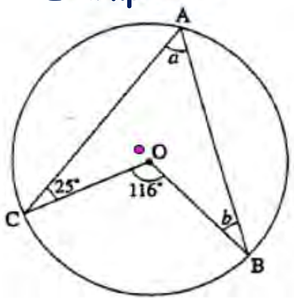
Example 5:



x = 180 – 30 – 90 = 40°

(Theorem 5 – angles on a tangent)

Example 4:



a = 88°

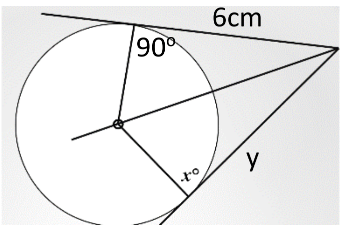
(Theorem 1 - angle at the centre)

To work out b:

● = 360 - 116 = 244° (angles around a point)

b = 360 - 244 - 25 - 88 = 3° (angles in a quadrilateral)

Example 6:



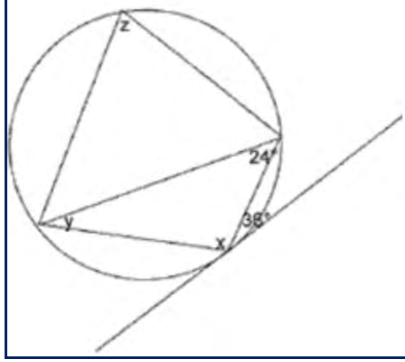
x = 90°

(Theorem 5 – angles on a tangent)

y = 6cm

(Theorem 6 – Two tangents)

Example 7:



y = 36°

(Theorem 6 - alternate segment)

x = 180 - 36 - 24 = 120°

(angles in a triangle)

z = 180 - 120 = 60°

(Theorem 4 - cyclic quadrilateral)

Mathematics

Higher

Unit 52

Algebraic Fractions

Adding and Subtracting Algebraic Fractions

When adding and subtracting algebraic fractions, do the same as you would if you were adding and subtracting fractions without algebra.

Step 1: Put in brackets where needed. Make the denominators the same by multiplying each fraction by the denominator of the other fraction. Make sure you do the same to the top and bottom.

Step 2: Add or subtract just the numerators so that you only have one denominator overall.

Step 3: Expand any brackets and simplify.

Example 1: Simplify $\frac{2}{x+3}+\frac{2}{3x+2}$

$\frac{\times(3x+2)}{\times(3x+2)}\ \frac{2}{x+3}+\frac{2}{3x+2}\ \frac{\times(x+3)}{\times(x+3)}$ ← Make the denominators the same.

$=\frac{2(3x+2)}{(x+3)(3x+2)}+\frac{2(x+3)}{(x+3)(3x+2)}$ ← Add just the numerators so that you have just one fraction (big line).

$=\frac{2(3x+2)+2(x+3)}{(x+3)(3x+2)}$ ← Expand the brackets.

$=\frac{6x+4+2x+6}{(x+3)(3x+2)}$ ← Simplify.

$=\frac{8x+10}{(x+3)(3x+2)}$

$=\frac{2(4x+5)}{(x+3)(3x+2)}$ ← Here you can factorise the top.

Example 2: Simplify $\frac{2}{x+1}-\frac{5}{4x+3}$

$\frac{\times(4x+3)}{\times(4x+3)}\ \frac{2}{x+1}-\frac{5}{4x+3}\ \frac{\times(x+1)}{\times(x+1)}$ ← Make the denominators the same.

$=\frac{2(4x+3)}{(x+1)(4x+3)}-\frac{5(x+1)}{(x+1)(4x+3)}$ ← Subtract just the numerators so that you have just one fraction (big line).

$=\frac{2(4x+3)-5(x+1)}{(x+1)(4x+3)}$ ← Expand the brackets. Be careful with the negative signs. Remember, you are multiplying the second bracket by -5 not 5.

$=\frac{8x+6-5x-5}{(x+1)(4x+3)}$ ← Simplify.

$=\frac{3x+1}{(x+1)(4x+3)}$ ← Here you cannot factorise the top, so leave it like this.

Mathematics

Higher

Unit 52

Simplifying Algebraic Fractions

Step 1: Factorise the numerator and denominator separately.

Step 2: Put them back as a fraction. Cancel out any brackets which are the same in both the numerator and denominator.

Example 1: Simplify $\frac{x^2-3x-28}{x^2-15x+56}$

Numerator:

$x^2-3x-28$

We want 2 numbers which multiply to give -28 and add to give -3.

These pairs multiply to give -28:

-1 x 28	1 x -28
-2 x 14	2 x -14
-4 x 7	4 x -7

Add to give -3

So, $x^2-3x-28=(x+4)(x-7)$

Denominator:

$x^2-15x+56$

We want 2 numbers which multiply to give 56 and add to give -15 (both negative).

These pairs multiply to give 56:

-1 x -56
-2 x -28
-4 x -14
-7 x -8

Add to give -15

So, $x^2-15x+56=(x-7)(x-8)$

Therefore, $\frac{x^2-3x-28}{x^2-15x+56}=\frac{(x+4)\cancel{(x-7)}}{\cancel{(x-7)}(x-8)}$

$=\frac{x+4}{x-8}$

$(x-7)$ appears in both the numerator and the denominator. These cancel out.

Example 2: Simplify $\frac{x^2-64}{3x^2-20x-32}$

Numerator:

x^2-64

Difference of two squares!

Square root each term and make sure the signs in the brackets are different.

So, $x^2-64=(x+8)(x-8)$

Denominator:

$3x^2-20x-32$

We want 2 numbers which multiply to give 3 x -32 = -96 and add to give -20.

These pairs multiply to give -96:

-1 x 96	1 x -96
-2 x 48	2 x -48
-3 x 32	3 x -32
-4 x 24	4 x -24
-6 x 16	6 x -16
-8 x 12	8 x -12

Add to give -20

So, $3x^2-20x-32=(3x+4)\frac{(3x-24)}{3}$

$=(3x+4)(x-8)$

Don't forget the 3 in front of the x^2. Check to see if you can divide any of the brackets. Here, we can divide $(3x-24)$ by 3.

Therefore, $\frac{x^2-64}{3x^2-20x-32}=\frac{(x+8)\cancel{(x-8)}}{(3x+4)\cancel{(x-8)}}$

$=\frac{x+8}{3x+4}$

$(x-8)$ appears in both the numerator and the denominator. These cancel out.

Mathematics

Curved Algebraic Graphs

Higher

Unit 53

Drawing Curves From their Equations

The equation of a curve is a way of expressing the relationship between the x-coordinates and the y-coordinates that lie on that curve.

Example: $y = x^2 + 3x - 9$

This says that the relationship between all the x-coordinates and all the y-coordinates is: "take the x-coordinate, square it, add on three lots of the x-coordinate, subtract 9, and you get the y-coordinate".

So, if a pair of coordinates such as (2, 1) has this relationship then it lies on the curve. If it does not, such as (5, 4), then it does not lie on the curve.

What you end up with is a curve that goes through all the co-ordinates which share that relationship

Method:

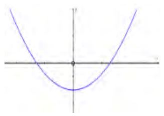
Step 1: If you are not given values of x to use then choose sensible values of x, ones that are small enough to fit on the paper, and easy enough to work out.

Step 2: Substitute these into the equation to get your y values.

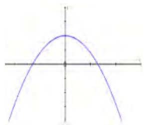
Step 3: Plot the points and join them up with a smooth curve (your pencil should not leave the paper, drawing one continuous curved line)

Check: If the equation has a positive x^2 term, then the graph would have a ∪ shape. If the equation has a negative x^2 term, then the graph would have an ∩ shape.

Eg: $2x^2 + 4x - 3$
Positive x^2 term,
so ∪ shape



Eg: $-x^2 + 5x$
Negative x^2 term,
so ∩ shape



Note: Be careful when substituting negative numbers.

Note: Pick $x = 0$ as one of your points, as it often makes it easier to work out the corresponding y value.

Substituting Numbers without a Calculator

If you are asked to draw a curve on a non-calculator paper then remember:

1. What order you must do operations – remember BIDMAS/BODMAS
2. The rules of negative numbers

Example: Substitute $x = -2$ into $y = x^2 - 4x + 2$

Replace the x terms with -2: $y = (-2)^2 - 4 \times -2 + 2$

Remembering BIDMAS/BODMAS do the squared term first: $y = 4 - 4 \times -2 + 2$

Then the multiplication: $y = 4 - -8 + 2$

The two minus signs together make a plus: $y = 4 + 8 + 2$

Work out the final answer: $y = 14$

So, the point you need to plot has the co-ordinates **(-2, 14)**

Substituting Numbers with a Calculator

Remember:

1. Put your negative numbers in brackets
2. Always do each calculation twice to make sure you did not press a wrong button.

Mathematics

Higher

Unit 53

Example 1:

$y = 2x^2 - 5x$

x	-2	-1	0	1	2	3	4
y		7	0	-3	-2		12

a) Complete the table above.

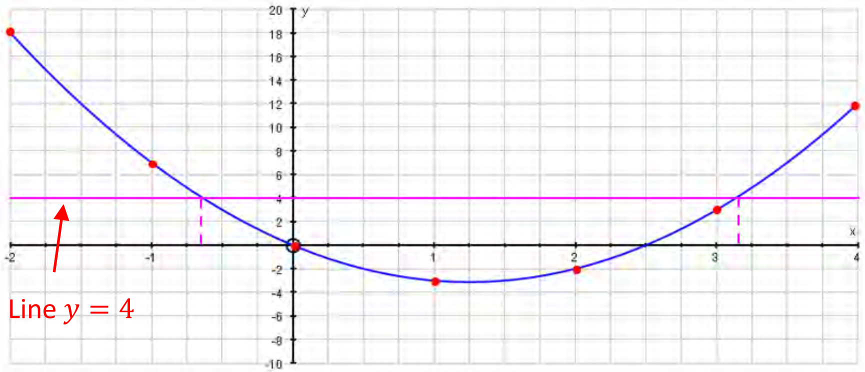
When $x = -2$, $y = 2 \times (-2)^2 - 5 \times (-2) = 18$

When $x = 3$, $y = 2 \times 3^2 - 5 \times 3 = 3$

b) Draw the graph of $y = 2x^2 - 5x$

$y = 2x^2 - 5x$

x	-2	-1	0	1	2	3	4
y	18	7	0	-3	-2	3	12



c) Draw the line $y = 4$ on the graph. Write down the values of x where the line $y = 4$ cuts the curve $y = x^2 - 3x - 4$.

(Where the line crosses the curve, read the corresponding x values)

Values of x are -0.7 and 3.2

Example 2:

$y = x^2 - 3x - 4$

x	-2	-1	0	1	2	3	4	5
y	6	0	-4	-6	-6	-4	0	6

a) Complete the table above. The top line of the table represents the values for x. These need to be substituted into the equation to work out y.

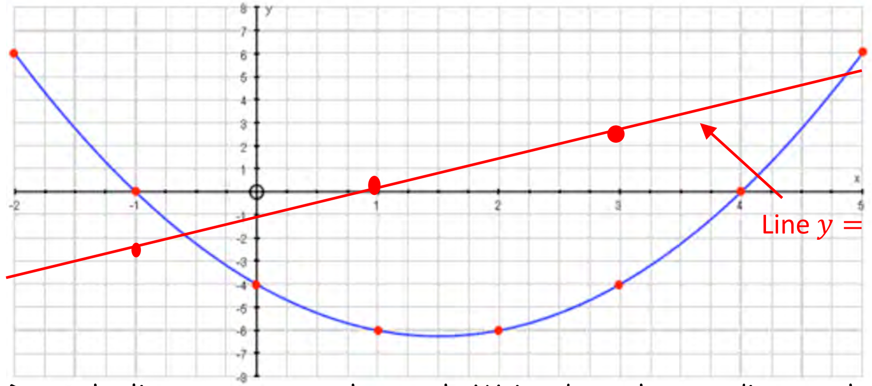
When $x = 0$, $y = 0^2 - 3 \times 0 - 4 = -4$

When $x = 5$, $y = 5^2 - 3 \times 5 - 4 = 6$

b) Draw the graph of $x^2 - 3x - 4$.

$y = x^2 - 3x - 4$

x	-2	-1	0	1	2	3	4	5
y	6	0	-4	-6	-6	-4	0	6



b) Draw the line $y = x - 2$ on the graph. Write down the coordinates where the line $y = x - 2$ cuts the curve $y = x^2 - 3x - 4$.

Substitute values of x into $y = x - 2$ in order to plot points and draw the line. Use a minimum of 3 values. E.g. when x = -2, x = 1 and x = 3. These should make a straight line.

For x = -2, y = -2 -2 = -4 Plot (-2, -4) For x = 1, y = 1 -2 = -1 Plot (1, -1)

For x = 3, y = 3 -2 = 1 Plot (3, 1)

Coordinates where the line crosses the curve are approximately (-0.5, -2) and (4.8, 6).

Mathematics

Higher

Unit 53

Finding the Gradient of Curved Graphs

To find the gradient of the line, we first draw a tangent at the point given.

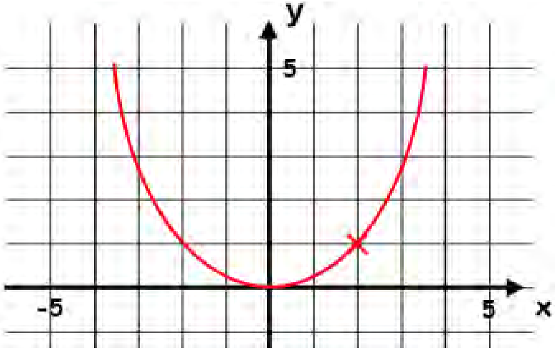
Then, use the formula:

$gradient = \frac{change\ in\ y}{change\ in\ x}$ or $gradient = \frac{y2-y1}{x2-x1}$

Where y_2 and y_1 are the coordinates on the y-axis.

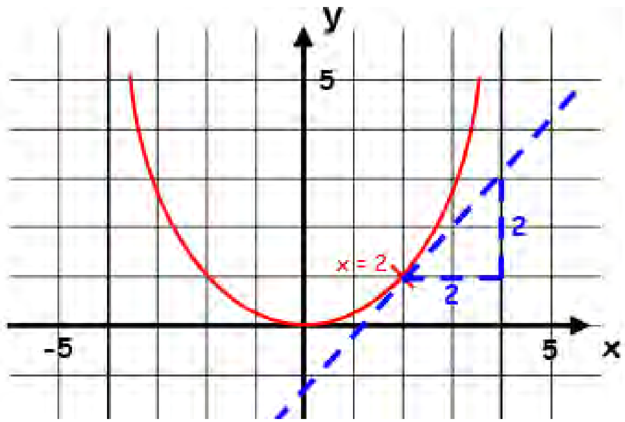
Where x_2 and x_1 are the coordinates on the x-axis.

Example: Find the gradient of the curve at the point $x = 2$.



Draw a straight line that just touches the curve where x = 2

- This line is known as a tangent to the curve
- You can calculate the gradient of it like on a straight line graph
- The value will be an estimate of the gradient of the curve at the given point



$$Gradient = \frac{change\ in\ y}{change\ in\ x}\ or\ \frac{y_2 - y_1}{x_2 - x_1}$$

$$Gradient = \frac{2}{2}$$

$$Gradient = 1$$

In this case, the gradient is positive as the tangent is going UP.

Mathematics

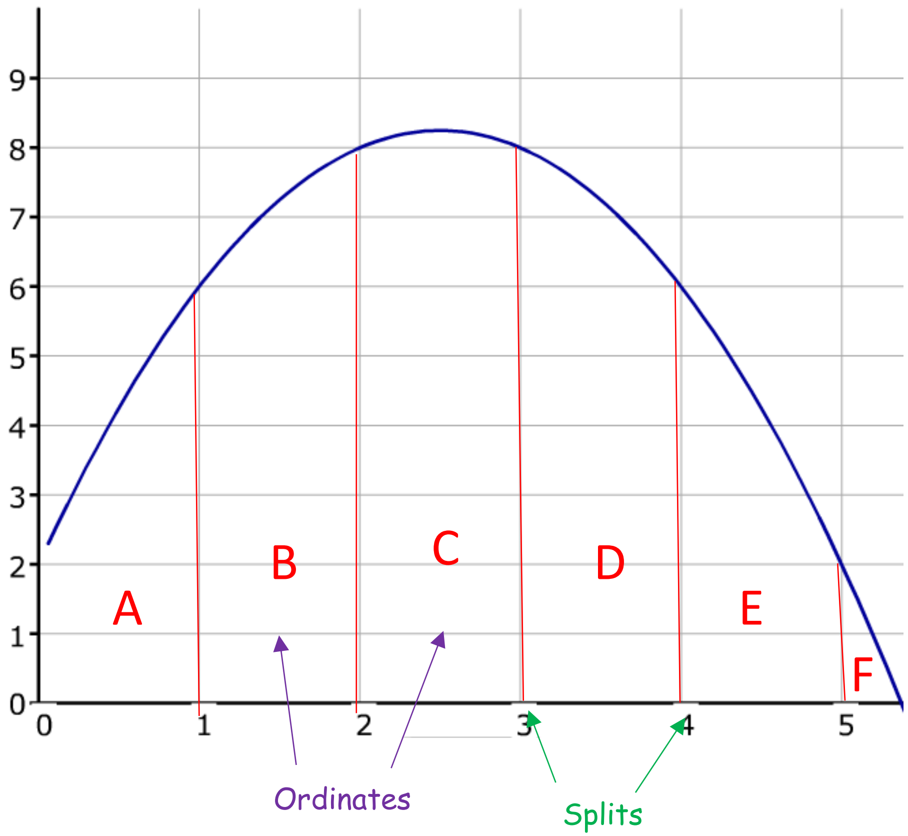
Higher

Unit 53

Finding the Area of Curved Graphs – The Trapezium Rule

We use the Trapezium Rule to find the area under graphs. For a curved graph, this will be an estimate of the area as the lines are not straight. It is called the trapezium rule as we split the curve up into trapeziums (and sometimes some triangles).

Example: Use the trapezium rule to find the area under the curve with **6 ordinates (areas)** or with **5 splits (lines between each area)**. The splits should each have an **equal width**.



Step 1: Split the area up into equal sections (you'll be told how many). Label these A, B, C, etc.

Step 2: Find the area of each section. Remember the formulae you need for area:

Area of a triangle = $\frac{1}{2}$ base x height

Area of trapezium = $\frac{1}{2}$ (a + b) h (where a and b are parallel and h is the perpendicular height)

Step 3: Add the areas together to find the total area.

$$Area\ A = \frac{1}{2} \times (a+b) \times h = \frac{1}{2} \times (2+6) \times 1 = 4$$

$$Area\ B = \frac{1}{2} \times (a+b) \times h = \frac{1}{2} \times (6+8) \times 1 = 7$$

$$Area\ C = \frac{1}{2} \times (a+b) \times h = \frac{1}{2} \times (8+8) \times 1 = 8$$

$$Area\ D = \frac{1}{2} \times (a+b) \times h = \frac{1}{2} \times (8+6) \times 1 = 7$$

$$Area\ E = \frac{1}{2} \times (a+b) \times h = \frac{1}{2} \times (6+2) \times 1 = 4$$

$$Area\ F = \frac{1}{2} \times b \times h = \frac{1}{2} \times 2 \times 0.5 = 0.5$$

Total area = 4 + 7 + 8 + 7 + 4 + 0.5 = 26.5

Trigonometric Graphs

Mathematics

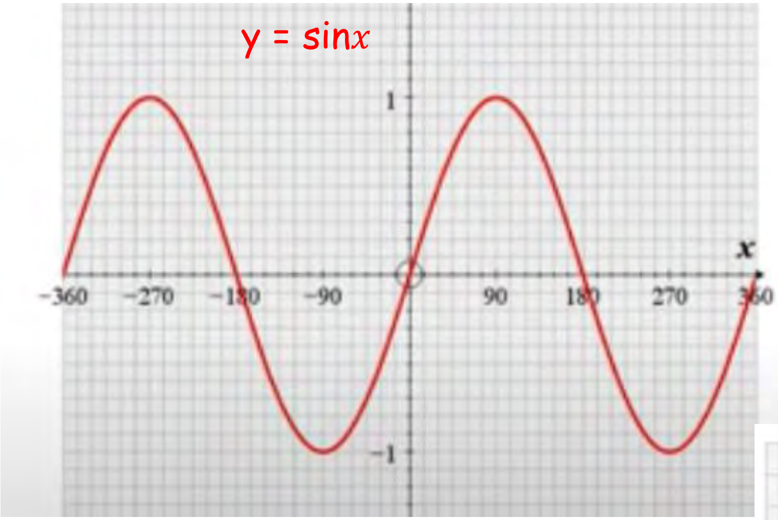
Higher

Unit 54

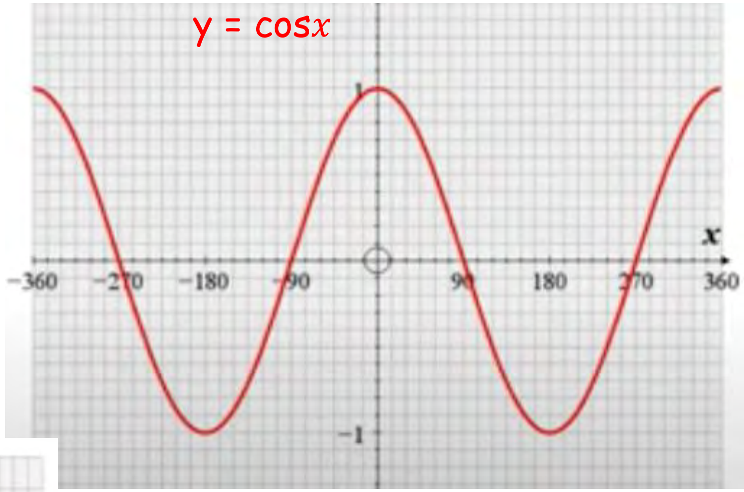
All these graphs have a pattern which repeats.

You need to learn these graphs and the important points.

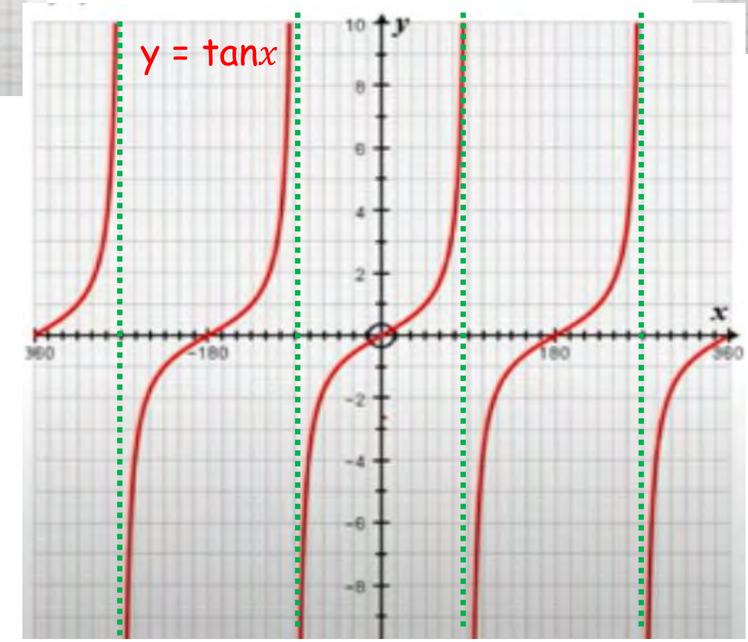
Sine graph



Cosine graph



Tan graph



The tan graph has asymptotes at $x = -270$, $x = -90$, $x = 90$, and $x = 270$. This means the graph goes off to infinity.

Mathematics

Higher

Unit 54

Solving Equations Using Trigonometric Graphs

Example 1: Solve $sinx = 0.5$ for values of x between –360° and 360°.

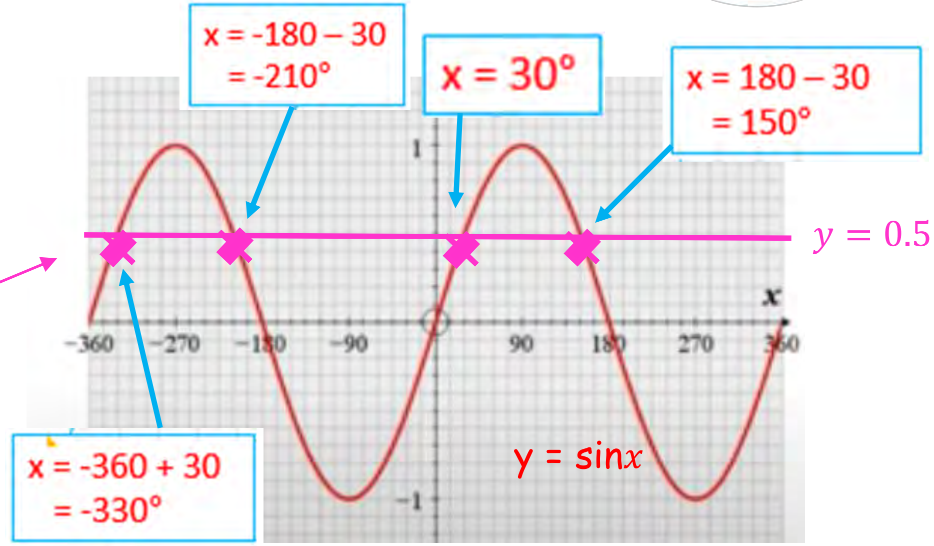
Rearrange the equation to get it in terms of x. $sinx = 0.5$

$$x = \sin^{-1}(0.5)$$

Type this into your calculator. $x = 30°$

Draw the line $y = 0.5$ onto your graph. If you look at the graph, the line at 0.5 crosses the curve at 4 points. Therefore, there should be 4 answers. Use the symmetry of the graph to work out the other 3.

The values of x that lie between –360° and 360° are 30°, 150°, -210° and -330°.



Example 2: Solve $3sinx = -2$ for values of x between –360° and 360°.

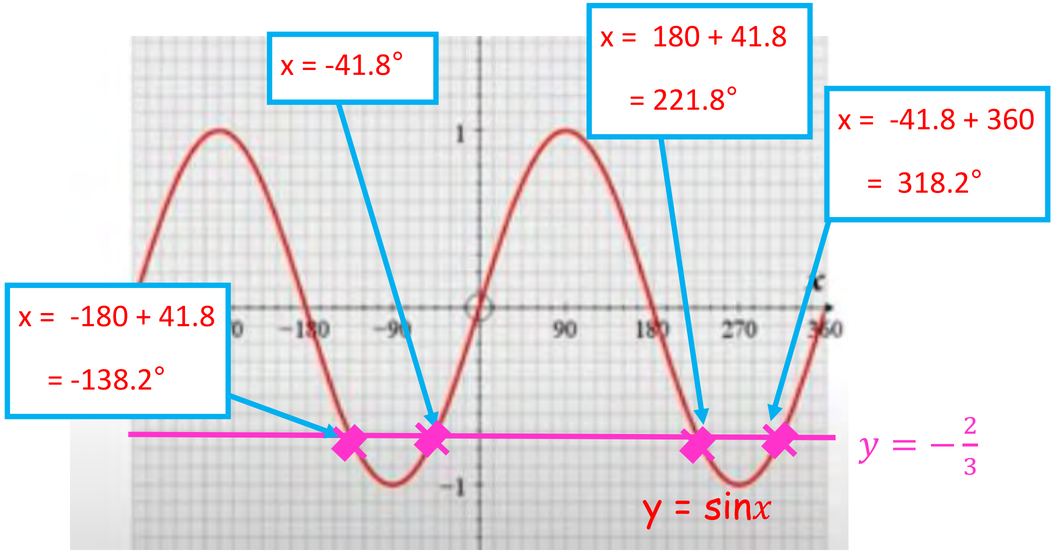
Rearrange the equation to get it in terms of x. $sinx = -\frac{2}{3}$

$$x = \sin^{-1}(-\tfrac{2}{3})$$

Type this into your calculator. $x = -41.8°$

Sketch the line $y = -\frac{2}{3}$ onto your graph. It doesn't need to be accurate. If you look at the graph, the line crosses the curve at 4 points so there should be 4 answers. Use the symmetry of the graph to work out the other 3.

The value of x that lie between –360° and 360° are -138.2°, -41.8°, 211.8° and 318.2°.



Mathematics

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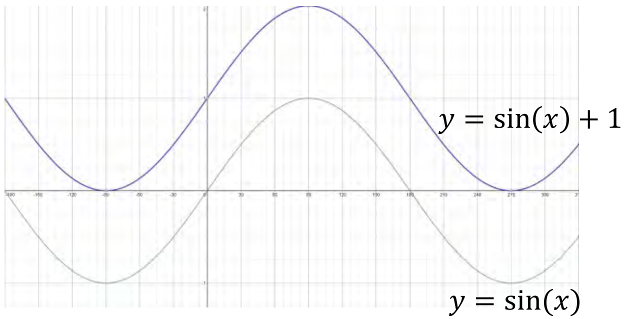
Transformations of Trigonometric Graphs

All graphs can be transformed by applying different rules to their original function $y = f(x)$

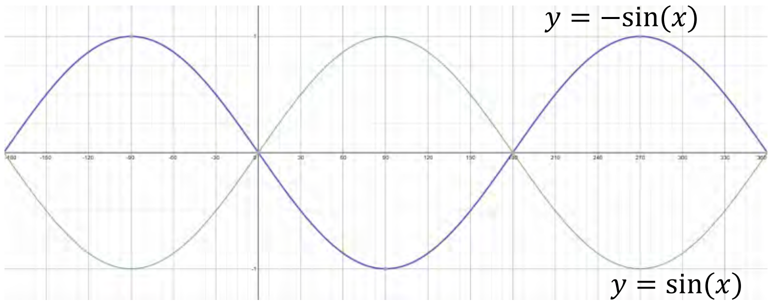
- $\boldsymbol{y = -f(x)}$ This will reflect a function in the *x* axis.
- $\boldsymbol{y = f(-x)}$ This will reflect a function in the *y* axis.
- $\boldsymbol{y = f(x) \pm a}$ This will move the graph up or down.
- $\boldsymbol{y = f(x \pm a)}$ This will move left for positive and right for negative.

Example 1:

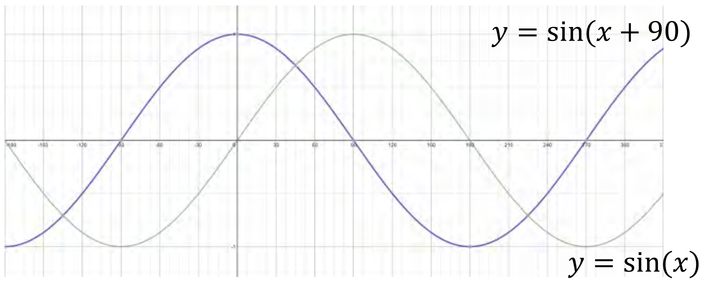
a) $y = \sin(x) + 1$



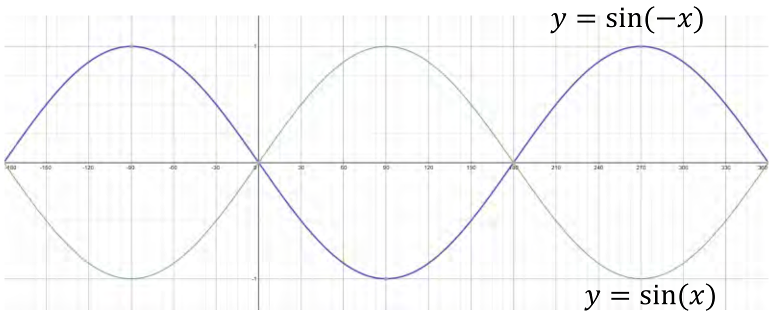
b) $y = -\sin(x)$



c) $y = \sin(x + 90)$



d) $y = \sin(-x)$



Mathematics

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Unit 55

Rational and Irrational Numbers

Rational Numbers are numbers which can be written as fractions (e.g. whole numbers, fractions, decimals such as 0.2 = $^1/_5$, $0.\dot{3}$ = $^1/_3$ and roots of square numbers such as $\sqrt{4}$ = 2).

Irrational Numbers are numbers which CANNOT be written as a fraction (e.g. π, $\sqrt{2}$, $\sqrt{5}$). When written as decimals, these numbers go on forever.

Surds – surds are irrational numbers which have square roots in them (so $\sqrt{2}$ is a surd but π is not). The square root of any number which is not a square number is a surd.

Surds

Rule 1

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

If you have a surd and you multiply it by another surd, then the answer is just the same as the surd of the original two numbers (a and b) multiplied together

e.g.

$$\sqrt{7} \times \sqrt{5} = \sqrt{7 \times 5} = \sqrt{35}$$

Rule 2

$$\sqrt{a} \times \sqrt{a} = a$$

If you multiply a surd by itself, then the answer is just the original number before it was square-rooted

e.g.

$$\sqrt{8} \times \sqrt{8} = \sqrt{8 \times 8} = \sqrt{64} = 8$$

Rule 3

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

If you divide a surd by another surd, then this is the same as the surd of the original numbers divided.

e.g.

$$\frac{\sqrt{28}}{\sqrt{2}} = \sqrt{\frac{28}{2}} = \sqrt{14}$$

Mathematics

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Simplifying Single Surds

You need to make the number under the square root sign as small as possible

Method

1. Split up the number being square-rooted into a product of at least one square number

2. Use Rule 1 to simplify your answer

Remember: Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100…

Example 1: Simplify $\sqrt{50}$

Split up 50. We ask ourselves: "which square number is a factor of 50?" = 25

$$50 = 25 \times 2$$

HINT: Put the square number first.

So, using Rule 1:

$$\sqrt{50} = \sqrt{25} \times \sqrt{2}$$

We know that $\sqrt{25}$ = 5

$$= 5\sqrt{2}$$

Invisible x sign. No need to write it in.

Example 2: Simplify $\sqrt{45}$

Split up 45. We ask ourselves: "which square number is a factor of 45?" = 9

$$45 = 9 \times 5$$

HINT: Put the square number first.

So, using Rule 1:

$$\sqrt{45} = \sqrt{9} \times \sqrt{5}$$

We know that $\sqrt{9}$ = 3

$$= 3\sqrt{5}$$

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Simplifying More Than One Surd (Multiplying)

Example: Simplify $\sqrt{\mathbf{90}} \times \sqrt{\mathbf{20}}$

Let's deal with each surd individually and split them up exactly like we did in the previous section:

$$90 = 9 \times 10$$
$$\sqrt{90} = \sqrt{9} \times \sqrt{10}$$
$$\sqrt{90} = 3 \times \sqrt{10} = 3\sqrt{10}$$

$$20 = 4 \times 5$$
$$\sqrt{20} = \sqrt{4} \times \sqrt{5}$$
$$\sqrt{20} = 2 \times \sqrt{5} = 2\sqrt{5}$$

So: $\sqrt{90} \times \sqrt{20} = 3\sqrt{10} \times 2\sqrt{5}$

To simplify further we multiply our whole numbers and our surds separately

$3 \times 2 = 6$ and $\sqrt{10} \times \sqrt{5} = \sqrt{50}$

So: $3\sqrt{10} \times 2\sqrt{5} = 6\sqrt{50}$

And if you wanted to be really clever, we can simplify even further

$$\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$$

So: $6\sqrt{50} = 6 \times 5\sqrt{2} = 30\sqrt{2}$

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There is more than one way of simplifying surds when **multiplying**.

Other methods:

Example 1: Simplify $\sqrt{90} \times \sqrt{20}$

Use rule 1

$= \sqrt{90 \times 20}$

$= \sqrt{1800}$

$= \sqrt{100 \times 18}$

$= \sqrt{100 \times 9 \times 2}$

$= \sqrt{100} \times \sqrt{9} \times \sqrt{2}$

$= 10 \times 3 \times \sqrt{2}$

$= 30\sqrt{2}$

Look for square numbers which go into 1800.

Example 2: Simplify $\sqrt{90} \times \sqrt{20}$

I noticed that 10 goes into both 90 and 20.

$= \sqrt{9 \times 10} \times \sqrt{2 \times 10}$

$= \sqrt{9} \times \sqrt{10} \times \sqrt{2} \times \sqrt{10}$

$= \sqrt{9} \times \sqrt{10} \times \sqrt{10} \times \sqrt{2}$

$= 3 \times 10 \times \sqrt{2}$

$= 30\sqrt{2}$

Changed the order to get the $\sqrt{10}$'s together.

Rule 2 says that $\sqrt{10} \times \sqrt{10} = 10$

Simplifying More Than One Surd (Dividing)

Good News: Do these in exactly the same way

Example: Simplify $\frac{\sqrt{60} \times \sqrt{20}}{\sqrt{12}}$

$= \frac{\sqrt{60 \times 20}}{\sqrt{12}}$

Use rule 1

$= \frac{\sqrt{120}}{\sqrt{12}}$

$= \sqrt{\frac{120}{12}}$

Use rule 3

$= \sqrt{10}$

This is the final answer. Even though you have probably noticed that 5 and 2 go into 10, neither 5 nor 2 are square numbers so we wouldn't be able to simplify it any more.

Mathematics

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Unit 55

Simplifying More Than One Surd (Adding and Subtracting)

We can only add and subtract surds of the same type

So, we must use our simplifying skills to change them into the same type.

$4\sqrt{3} + 5\sqrt{3} = 9\sqrt{3}$ (in the same way we would do 4a + 5a = 9a)

$10\sqrt{5} - 3\sqrt{5} = 7\sqrt{5}$

$2\sqrt{7} + 8\sqrt{6}$ We can't simplify this because the numbers under the roots are different (in the same way we can't simplify 2a + 8b).

Example 1: Simplify $\sqrt{12} + \sqrt{27}$

The answer is definitely NOT: $\sqrt{39}$

We need to simplify the surds first:

$12 = 4 \times 3$

$\sqrt{12} = \sqrt{4} \times \sqrt{3}$

$\sqrt{12} = 2 \times \sqrt{3} = 2\sqrt{3}$

$27 = 9 \times 3$

$\sqrt{27} = \sqrt{9} \times \sqrt{3}$

$\sqrt{27} = 3 \times \sqrt{3} = 3\sqrt{3}$

So: $\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3}$

Our surds are now of the same type. Each term has $\sqrt{3}$ in it.

We can now just add our whole numbers. $2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$

Example 2: Simplify $\sqrt{63} - \sqrt{28}$

Simplify the surds:

$63 = 9 \times 7$

$\sqrt{63} = \sqrt{9} \times \sqrt{7}$

$\sqrt{63} = 3 \times \sqrt{7} = 3\sqrt{7}$

$28 = 4 \times 7$

$\sqrt{28} = \sqrt{4} \times \sqrt{7}$

$\sqrt{28} = 2 \times \sqrt{7} = 2\sqrt{7}$

So: $\sqrt{63} - \sqrt{28} = 3\sqrt{7} - 2\sqrt{7}$

Our surds are now of the same type. Each term has $\sqrt{7}$ in it.

We can now just subtract our whole numbers. $3\sqrt{7} - 2\sqrt{7} = \sqrt{7}$

Mathematics

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Expanding Brackets with Surds

Rule: Use FOIL to multiply out the brackets as you would in algebra (multiply every term in the first bracket by every term in the second bracket).

Example 1: $(3+\sqrt{5})(6+\sqrt{5})$

First: 3 x 6 = 18

Outside: 3 x $\sqrt{5}$ = $3\sqrt{5}$

Inside: $\sqrt{5}$ x 6 = $6\sqrt{5}$

Last: $\sqrt{5}$ x $\sqrt{5}$ = 5

So: $(3+\sqrt{5})(6+\sqrt{5}) = 18 + 3\sqrt{5} + 6\sqrt{5} + 5$

= $23 + 9\sqrt{5}$

Simplifying the whole numbers gives

18 + 5 = 23

Simplifying the surds gives

$3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}$

(We can add here as the surds are the same)

Example 2: $(\sqrt{2}+7)(4-\sqrt{8})$

First: $\sqrt{2}$ x 4 = $4\sqrt{2}$

Outside: $\sqrt{2}$ x $-\sqrt{8}$ = $-\sqrt{16}$ = -4

Inside: 7 x 4 = 28

Last: 7 x $-\sqrt{8}$ = $-7\sqrt{8}$ = $-7\sqrt{4}$ x $\sqrt{2}$

= -7 x 2 x $\sqrt{2}$ = $-14\sqrt{2}$

So: $(\sqrt{2}+7)(4-\sqrt{8}) = 4\sqrt{2} - 4 + 28 - 14\sqrt{2}$

= $24 - 10\sqrt{2}$

Simplifying the whole numbers gives

-4 + 28 = 24

Simplifying the surds gives

$4\sqrt{2} - 14\sqrt{2} = -10\sqrt{2}$

(We can subtract here as the surds are the same)

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Rationalising the Denominator

What does 'Rationalising the Denominator' mean?

It is considered a bit untidy to have a surd on the bottom of a fraction (the denominator).

If we can get rid of all the surds off the bottom of a fraction, we get rid of all the irrational numbers, and so we **rationalise the denominator**.

Method:

Multiply the top and the bottom of the fraction by the surd which is on the bottom of the fraction.

Example - Single Surd: Rationalise the denominator of: $\frac{2}{\sqrt{3}}$

We don't like the look of the $\sqrt{3}$ on the bottom.

To rationalise the denominator we multiply by a value which is equivalent to 1 but contains the surd at the bottom.

This means we multiply by $\frac{\sqrt{3}}{\sqrt{3}}$.

Be careful: Remember, whatever we multiply the bottom of the fraction by, we must also multiply the top by.

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$2 \times \sqrt{3} = 2\sqrt{3}$$

And using Rule 2...

$$\sqrt{3} \times \sqrt{3} = 3$$

$$= \frac{2\sqrt{3}}{3}$$

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Changing Recurring/Repeating Decimals into Fractions

Rule

We want two decimals with the same recurring numbers after the decimal point. This way, when we subtract one number from the other, we get rid of any recurring or repeating numbers.

Example 1: Write $0.\dot{4}$ as a recurring decimal

$x = 0.\dot{4}$ — Label the number in the question as $x = 0.\dot{4}$

$10x = 4.\dot{4}$ — Multiply **both** sides of the '=' sign by **10** to get another decimal with the **same** recurring number after the decimal point.

$10x - x = 4.\dot{4} - 0.\dot{4}$

$9x = 4$

Subtract the first decimal from the second so that you have a number with no recurring decimal. Make sure you subtract the 'x's too.

$x = \frac{4}{9}$ — Solve the equation to find x. (Divide by 9)

Example 2: Write $0.\dot{1}\dot{2}$ as a recurring decimal

$x = 0.\dot{1}\dot{2}$ — Label the number in the question as $x = 0.\dot{1}\dot{2}$

$100x = 12.\dot{1}\dot{2}$ — This time multiply **both** sides of the '=' sign by **100** to get another decimal with the **same** recurring number after the decimal point.

$100x - x = 12.\dot{1}\dot{2} - 0.\dot{1}\dot{2}$

$99x = 12$

Subtract the first decimal from the second so that you have a number with no recurring decimal. Make sure you subtract the 'x's too.

$x = \frac{12}{99}$ — Solve the equation to find x. (Divide by 99)

$x = \frac{4}{33}$ — Simplify the fraction if it's possible.

Example 3: Write $0.1\dot{6}$ as a recurring decimal

$x = 0.1\dot{6}$ — Label the number in the question as $x = 0.1\dot{6}$

$10x = 1.\dot{6}$

$100x = 16.\dot{6}$

This time multiply **both** sides of the '=' sign by **10 AND by 100** to get two decimals with the **same** recurring number after the decimal point.

We cannot use $x = 0.1\dot{6}$ as the numbers after the decimal point are different to the numbers after the point for $10x$ and $100x$.

$100x - 10x = 16.\dot{6} - 1.\dot{6}$

$90x = 15$

Subtract the two **new decimals** so that you have a number with no recurring decimal.

$x = \frac{15}{90}$ — Solve the equation to find x. (Divide by 90)

$x = \frac{1}{6}$ — Simplify the fraction if it's possible.

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Graph Sketching

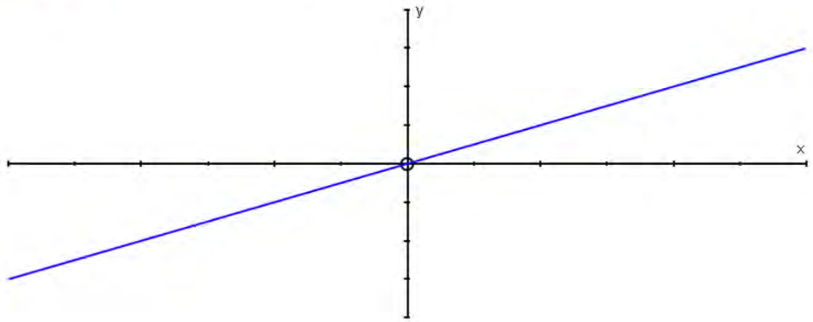
Shapes of Graphs

You can look at the equation of a graph and have a good idea of what shape it will be. This will help when drawing it. The following are general shapes of graphs which you should recognise.

1. Linear with a positive coefficient of x (a straight line going UP)

Equation: Highest power of x is 1, and the x term is positive

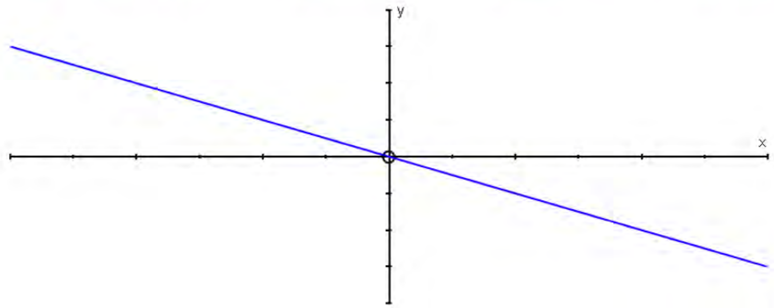
Examples: $y = 2x + 3$ $y = x - 8$ $y = 5x$ $y = 9x - 6$



2. Linear with negative coefficient of x (this is a straight line going DOWN)

Equation: Highest power of x is 1, and the x term is negative

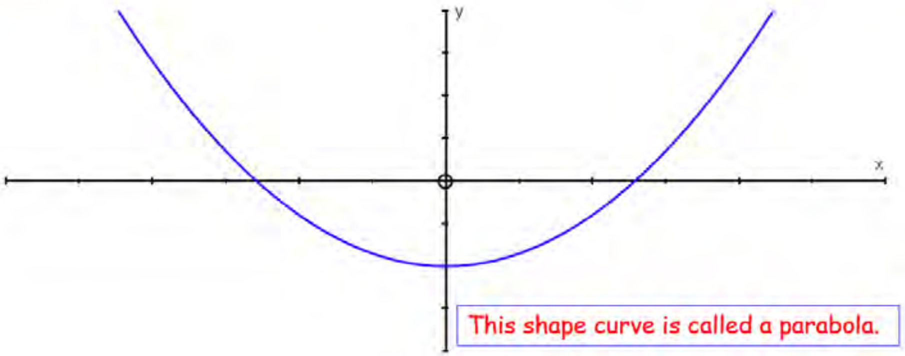
Examples: $y = -5x + 3$ $y = -x - 3$ $y = -7x$ $y = 5 - 6x$



3. Quadratic with a positive coefficient of x^2 (the curve looks like a smiley face)

Equation: Highest power of x is 2, and the x^2 term is positive

Examples: $y = x^2$ $y = x^2 + 5$ $y = x^2 - 3x + 2$ $y = 3x^2 + 2x - 6$

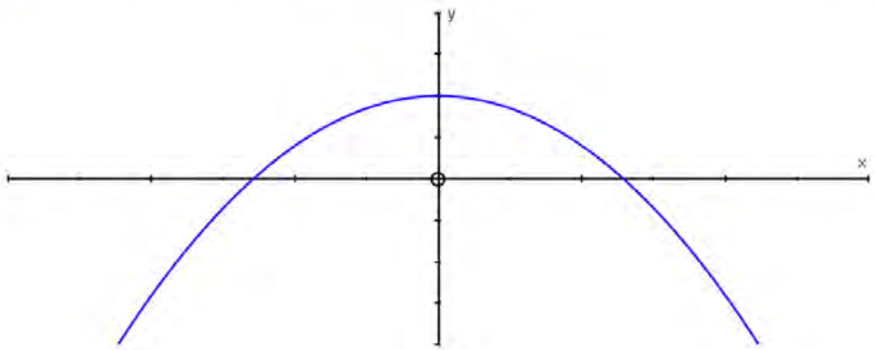


This shape curve is called a parabola.

4. Quadratic with a negative coefficient of x^2 (the curve looks like a sad face)

Equation: Highest power of x is 2, and the x^2 term is negative

Examples: $y = -x^2$ $y = -2x^2 + 4$ $y = -2(x^2 - 5x + 5)$ $y = 5 + 3x - x^2$



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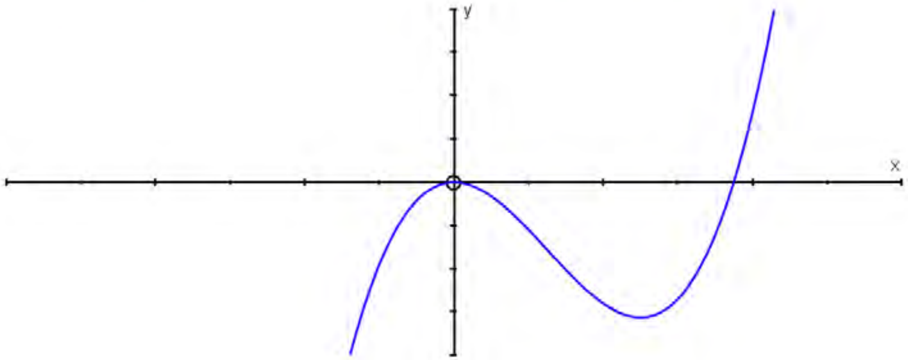
5. Cubic with a positive coefficient of x^3

Equation: Highest power of x is **3**, and the x^3 term is positive

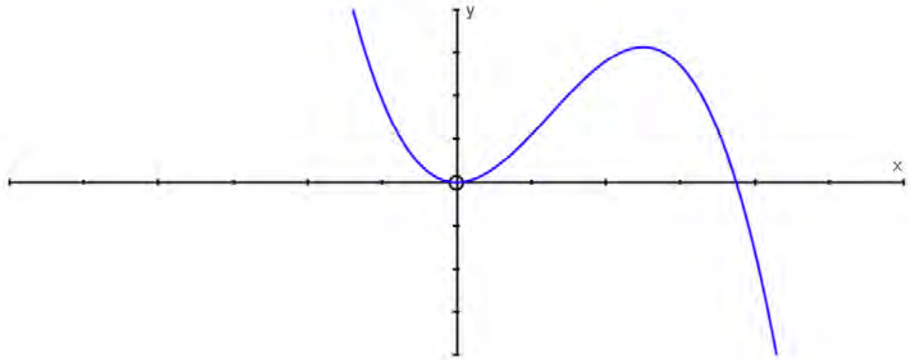
Examples: $y = x^3$ $\quad y = x^3 + 10$ $\quad y = x^3 - 2x^2 - 4x + 2$ $\quad y = 2x^3 - 6x$



6. Cubic with a negative coefficient of x^3

Equation: Highest power of x is **3**, and the x^3 term is negative

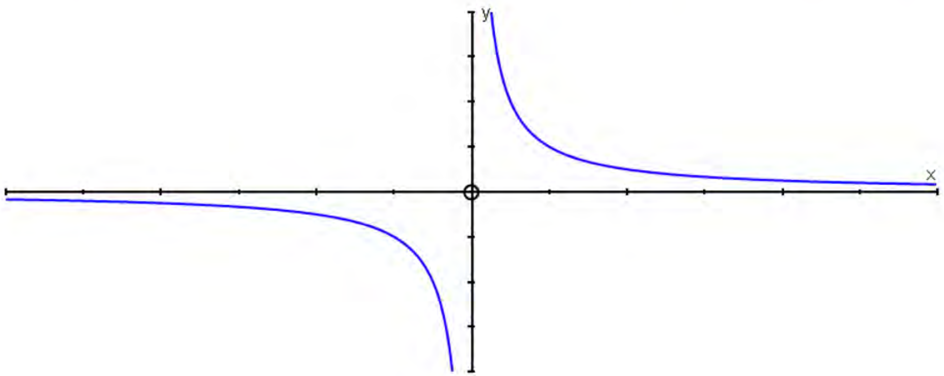
Examples: $y = -x^3$ $\quad y = -5x^3 + 2$ $\quad y = -4x^3 + x^2 + 5$ $\quad y = 5 + 3x + 5x^2 - x^3$



7. Positive Reciprocal

Equation: Contains a fraction with a positive x on the bottom

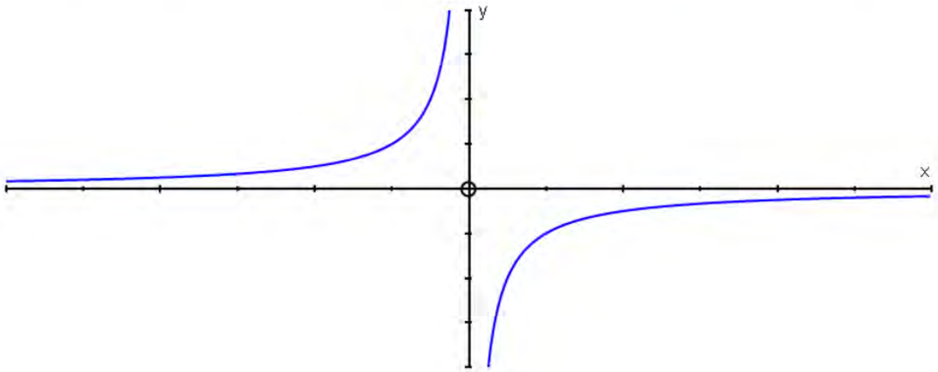
Examples: $y = \frac{1}{x}$ $\quad y = \frac{5}{x}$ $\quad y = \frac{7}{2x}$ $\quad y = \frac{3}{4x} + 2$



8. Negative Reciprocal

Equation: Contains a fraction with a negative x on the bottom

Examples: $y = -\frac{1}{x}$ $\quad y = \frac{5}{-x}$ $\quad y = -\frac{7}{2x}$ $\quad y = 2 - \frac{2}{x}$



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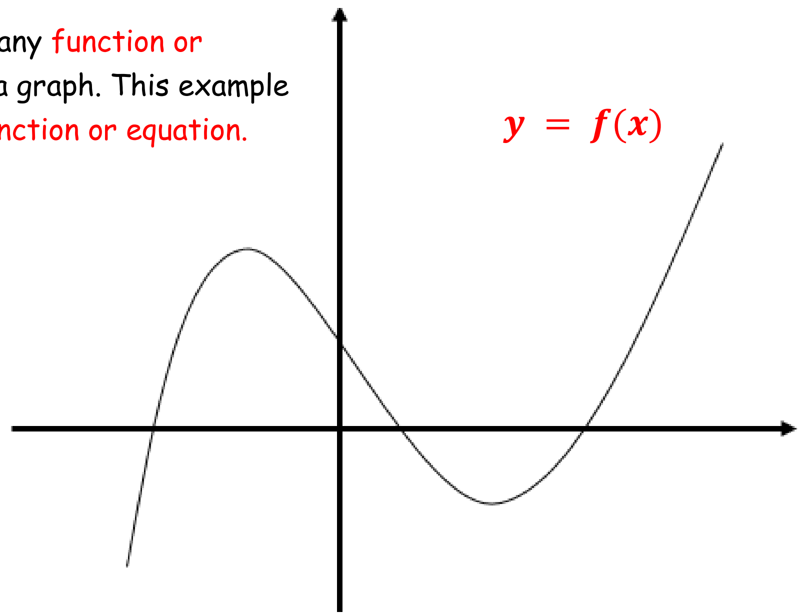
Transformations of Graphs

There are **FIVE** basic types of transformation of a graph.
i.e. five ways we can move the graph or change its shape.

Notation

$y = f(x)$ is any function or equation of a graph. This example is a cubic function or equation.



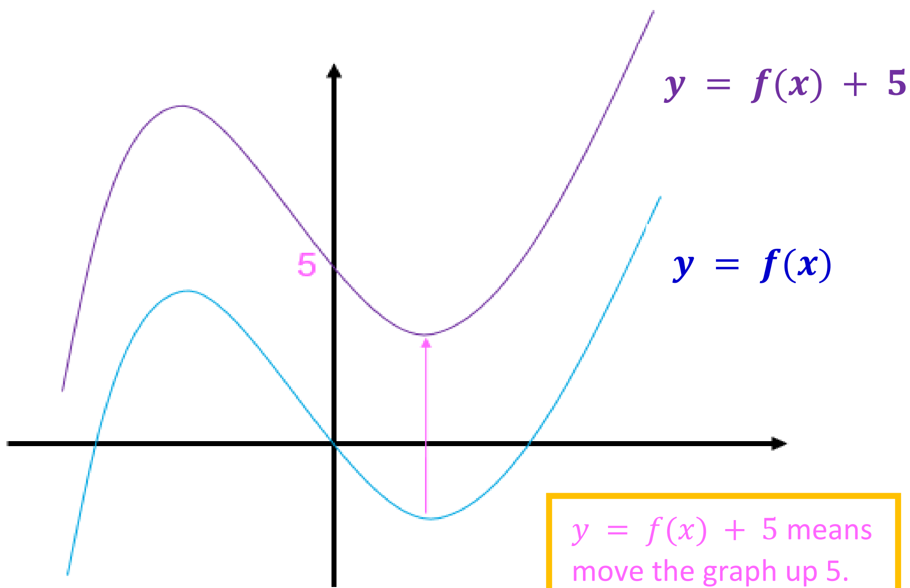
Rule 1: $y = f(x) + a$

This shifts the graph in the y-direction.

In other words:

For $y = f(x) + 2$ the graph goes UP BY 2

For $y = f(x) - 7$ the graph goes DOWN BY 7



$y = f(x) + 5$ means move the graph up 5.

Make sure you mark any important points on the axes.

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Rule 2: $y = f(x + a)$

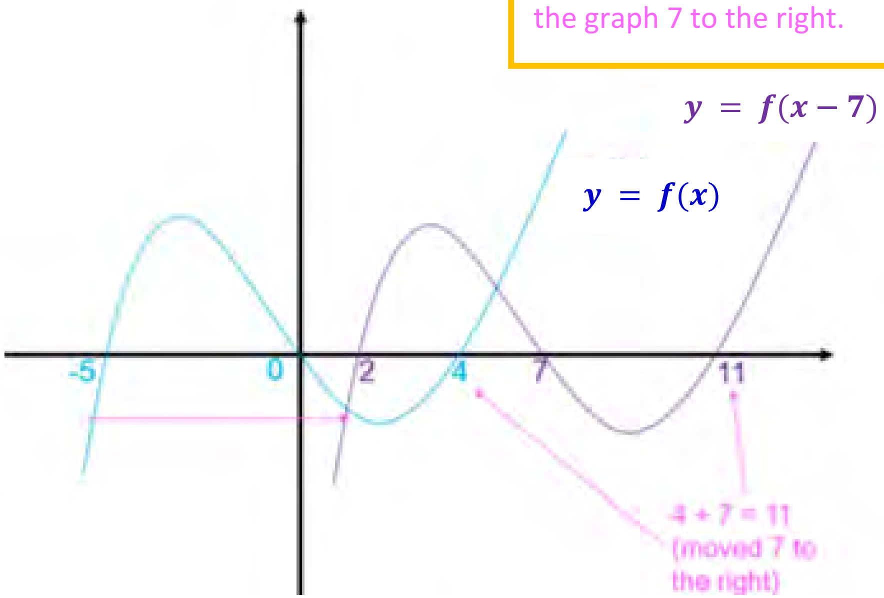
This shifts the graph in the x-direction.

In other words:

For $y = f(x + 2)$ the graph goes LEFT BY 2

For $y = f(x - 7)$ the graph goes RIGHT BY 7

$y = f(x - 7)$ means move the graph 7 to the right.

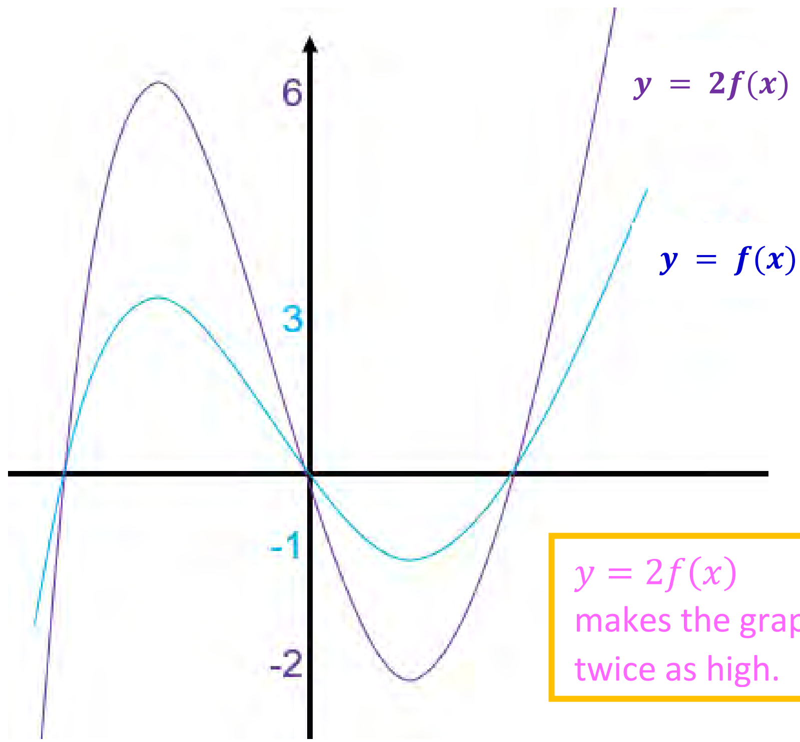


HINT: When the + and – is INSIDE the brackets, they have the opposite effect to what you would expect.
+ goes left and – goes right.

Rule 3: $y = af(x)$

This is a '**stretch**' in the **y-direction**, by a scale factor a.

Every y-coordinate is multiplied by a.



$y = 2f(x)$ makes the graph twice as high.

HINT: When drawing your new curve, the points which are on the x-axis stay the same.

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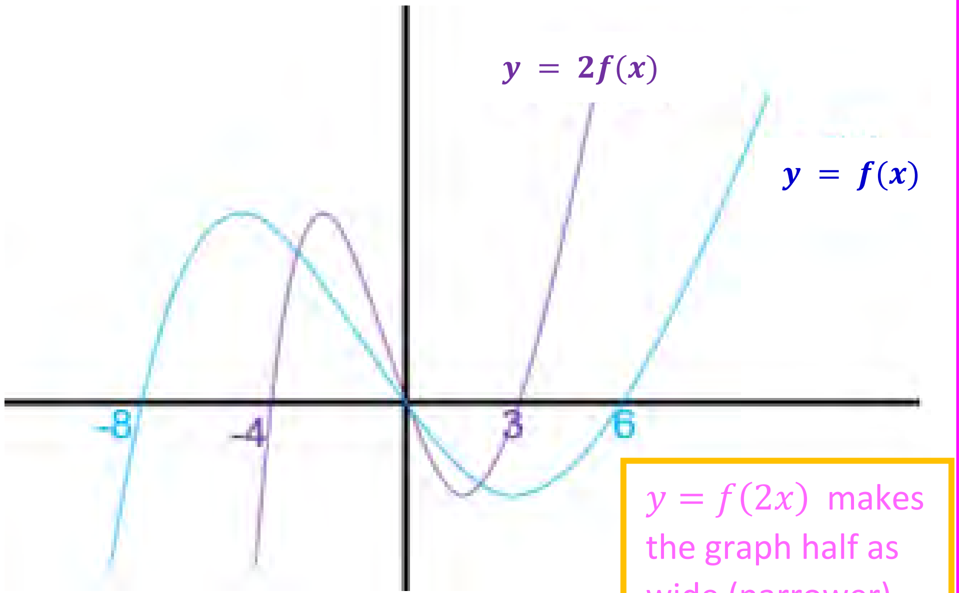
Rule 4: $y = f(ax)$

This is a '**stretch**' in the x**-direction**, by a scale factor $\frac{1}{a}$.

(Makes the curve narrower)

Every x-coordinate is multiplied by $\frac{1}{a}$ (or divided by a).

The y values stay the same.



$y = f(2x)$ makes the graph half as wide (narrower).

HINT: When drawing your new curve, the points which are on the y-axis stay the same.

Rule 5: $y = -f(x)$

This is a reflection (mirror image) in the x**-axis**.

Every x coordinate stays the same. Every y coordinate changes sign.

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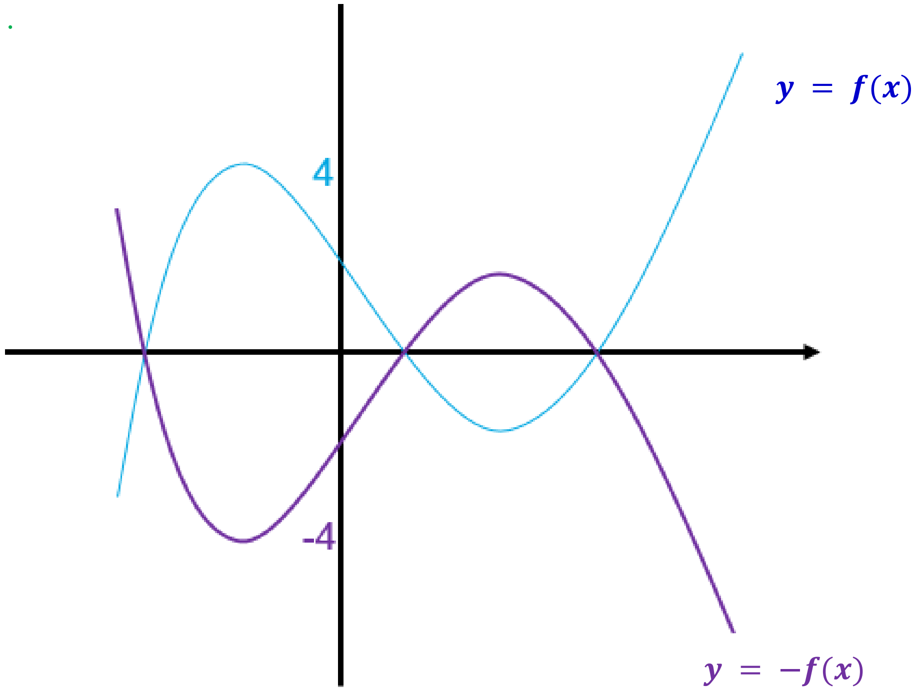

HINT: It sometimes helps to turn your pages sideways to make it easier to draw.

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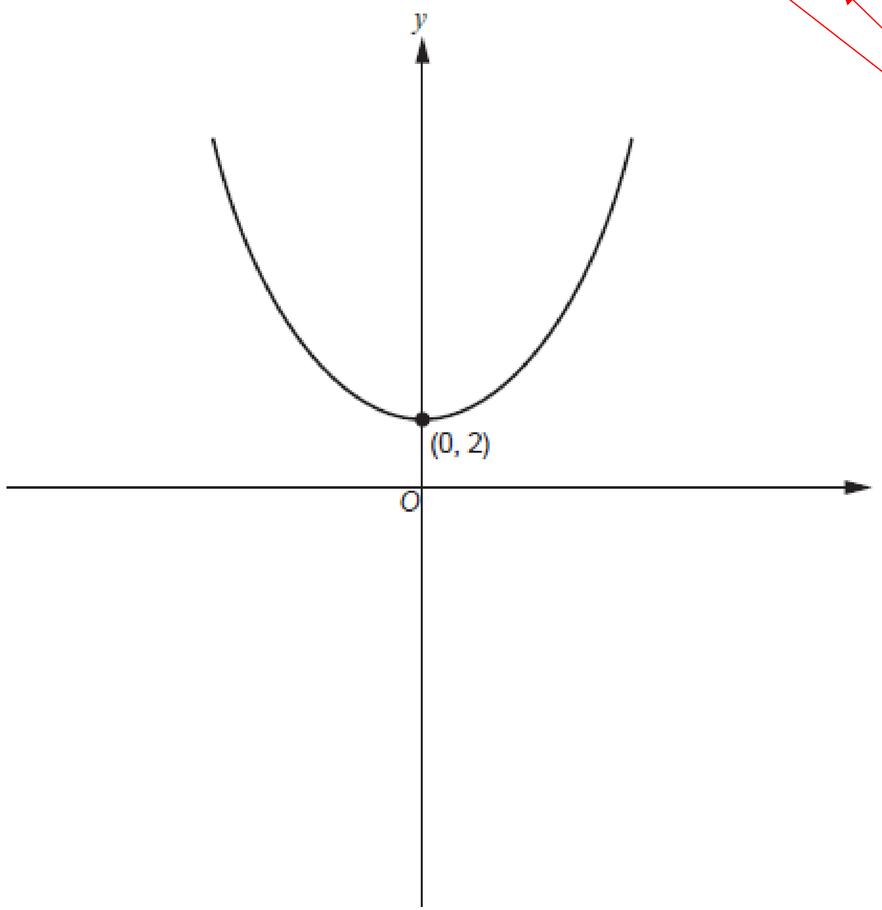
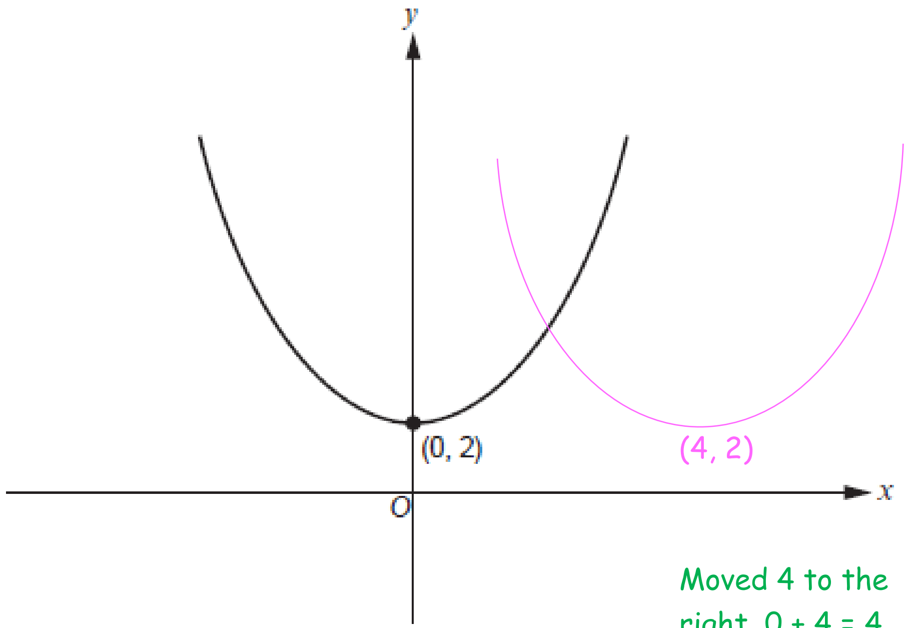
Example: GCSE Question.

The graph shows a sketch of the curve with equation $y = x^2 + 2$. The lowest point of the curve has coordinates $(0, 2)$.

On the same axes, sketch the graph of the curve with equation $y = (x - 4)^2 + 2$. Indicate clearly the coordinates of the lowest point on the new curve.

If we look at the original equation and the new one, the 'extra' bit is the -4. This is INSIDE the bracket, so we use RULE 2. Move the curve 4 to the right. Remember to mark any important points or coordinates.





Moved 4 to the right, 0 + 4 = 4.